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GENERALIZATION OF A RESULT FOR COSINE SERIES ON THE L^1 NORM

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ABSTRACT. A L^1-estimate will be established for cosine series, considering the generalized Fomin-class $\mathcal{F}_\varphi$, where φ is a function more general than the power function

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1. INTRODUCTION

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \tag{1.1}$$

Many authors ([1], [2], [9] – [15]) have investigated coefficient conditions guaranteeing that (1.1) is a Fourier series of some function $f \in L^1$ and they have given estimates for $\int_0^\pi |f(x)|dx$ via the sequence $\{a_n\}$.

Recently Z. Tomovski [15] proved a theorem of this type by using the class of coefficients defined by Fomin [1] as follows: a sequence $\{a_n\}$ belongs to $\mathcal{F}_p$ $(p > 1)$ if $a_k \to 0$ and

$$\sum_{k=1}^{\infty} \left\{ \frac{1}{k} \sum_{i=k}^{\infty} |\Delta a_i|^p \right\}^{\frac{1}{p}} < \infty. \tag{1.2}$$

Now we can formulate Z. Tomovski's result [15]:

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Theorem 1.1. *Let* $\{a_n\} \in \mathcal{F}_p,\ 1 < p \leq 2$, *then the series (1.1) is a Fourier series and the following inequality holds:*

$$\int_0^\pi |f(x)|dx \leq C_p \sum_{n=1}^{\infty} \left\{ \frac{1}{n} \sum_{k=n}^{\infty} |\Delta a_k|^p \right\}^{\frac{1}{p}}, \tag{1.3}$$

where C_p *depends only on* p.

Recently we ([7]) investigated the properties of classes of numerical sequences obtained by using functions more general than the power functions. Such functions were used first of all in the works of H.P. Mulholland [5] and M. Mateljevič and M. Pavlovič [8]. The following definition is due to Mateljevič and Pavlovič.

$\Delta(q,p)$ $(q \geq p > 0)$ denotes the family of the nonnegative real functions $\varphi(x)$ defined on $[0;\infty)$ with the following properties: $\varphi(0) = 0$, $\frac{\varphi(t)}{t^q}$ is nonincreasing and $\frac{\varphi(t)}{t^p}$ is nondecreasing on $(0;\infty)$. Δ will denote the set of the functions $\varphi(x) \in \Delta(q,p)$ for some $q \geq p > 0$.

Using this notion in [7] we defined the following class: a nullsequence $\{a_n\}$ belongs to the class $\mathcal{F}_\varphi$ for some $\varphi \in \Delta$ if

$$\sum_{n=1}^{\infty} \overline{\varphi} \left(\frac{1}{n} \sum_{k=n}^{\infty} \varphi(|\Delta a_k|) \right) < \infty, \tag{1.4}$$

where $\overline{\varphi}$ is the inverse of the function φ.

The aim of the present paper is to generalize Theorem 1.1 using $\mathcal{F}_\varphi$ instead of $\mathcal{F}_p$, where $\varphi \in \Delta$. Since our goal is to get a result concerning such functions as $\varphi(x) = x\ \log^\alpha(1+x)$ and not only for functions which are generalizations of x^p $(p > 1)$, we therefore need to define two subclasses of Δ. Namely we use the following definitions: $\Delta^{(1)}$ denotes the family of functions $\varphi(x)$ belonging to $\Delta(q,p)$ for some $q \geq p > 1$ and $\Delta^{(2)}$ is the collection of functions $\varphi(x)$ from $\Delta(q,1)$ for some $q > 1$ such that for all $A > 0$ there exists $p := p(A) > 1$ satisfying the condition that $\frac{\varphi(x)}{x^p}$ is nondecreasing on $(0;A)$. It is obvious that $\Delta^{(1)} \subset \Delta^{(2)}$.

After giving these definitions we can formulate our result which generalizes Theorem 1.1 (if $\varphi \in \Delta^{(1)}$). Furthermore, it contains the case like $\varphi(x) = x\ \log^\alpha(1+x)$ $(\alpha > 0)$.

2. Result

Theorem 2.1. *Let* $\{a_n\} \in \mathcal{F}_\varphi$ *for* $\varphi \in \Delta^{(2)}$. *Then the series (1.1) is a Fourier series and the following estimate holds*

$$\int_0^\pi |f(x)|dx \leq C_\varphi \sum_{n=1}^{\infty} \overline{\varphi} \left(\frac{1}{n} \sum_{k=n}^{\infty} \varphi(|\Delta a_k|) \right), \tag{2.1}$$

where C_φ *is a constant depending only on* φ.

Remark 2.2. In [3] and [4] L. Leindler investigated a relation among classes of numerical sequences other than $\mathcal{F}_p$ (see the classes denoted by $S_p,\ \mathcal{F}_p^*,\ S_p(\delta),\ S_p(A)$) and he proved that all these classes coincide. Later in [7] we defined the classes of sequences $S_\varphi,\ \mathcal{F}_\varphi^*,\ S_\varphi(\delta),\ S_\varphi(A)$ exchanging the functions x^p to $\varphi(x)$ and showed that all these classes also coincide. Therefore in Theorem 2.1 the class $\mathcal{F}_\varphi$ can be replaced by any of the above mentioned classes, if $\varphi \in \Delta^{(2)}$.

3. LEMMAS

Lemma 3.1. *([11]). Let the nullsequence $\{a_n\}$ be of bounded variation and*

$$\sum_{i=2}^{\infty}\sum_{k=1}^{[i/2]}\left|\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right|<\infty$$

then (1.1) is a Fourier series and the following estimate holds:

$$\int_0^{\pi}|f(x)|dx \le C\left(\sum_{k=0}^{\infty}|\Delta a_k|+\sum_{i=2}^{\infty}\left|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right|\right), \tag{3.1}$$

where C is some absolute constant.

Lemma 3.2. *([1]). Let $\{a_n\}$ be a nullsequence. Then the following estimate holds:*

$$\sum_{i=2}^{\infty}\sum_{k=1}^{[i/2]}\left|\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right| \le C_p\sum_{s=1}^{\infty}\Delta_s^{(p)}, \tag{3.2}$$

where

$$\Delta_s^{(p)}=\left\{\frac{1}{2^{s-1}}\sum_{k=2^{s-1}+1}^{2^s}|\Delta a_k|^p\right\}^{\frac{1}{p}}, \qquad 1<p\le 2,$$

and C_p is a constant depending only on p.

Lemma 3.3. *([5]). If $\frac{\Psi(x)}{x}$ is increasing then for all sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative numbers*

$$\Psi\left(\frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n a_i}\right)\le\frac{\sum_{i=1}^n a_i\Psi(2b_i)}{\sum_{i=1}^n a_i}$$

holds.

Lemma 3.4. *([6]). Let $\rho(x)$ denote a nonnegative function increasing to infinity such that $\frac{\rho(x)}{x}$ is decreasing to zero when x is increasing from zero to infinity. Furthermore, if $a_n\ge 0,\ \lambda_n>0$ for all n, then*

$$\sum_{n=1}^{\infty}\lambda_n\rho\left(\frac{a_n}{\lambda_n}\sum_{k=1}^{n}\lambda_k\right)\le C_\rho\sum_{n=1}^{\infty}\lambda_n\rho\left(\sum_{k=n}^{\infty}a_k\right), \tag{3.3}$$

where C_ρ depends only on the function $\rho(x)$.

Lemma 3.5. *Let $b_n\ge 0$ for all n and let $\overline{\varphi}$ be the inverse of the function $\varphi(x)\in\Delta^{(2)}$. Then*

$$\sum_{n=1}^{\infty}\overline{\varphi}(b_n)\le K_{\overline{\varphi}}\cdot\sum_{n=1}^{\infty}\overline{\varphi}\left(\frac{\sum_{k=n}^{\infty}b_k}{n}\right), \tag{3.4}$$

where $K_{\overline{\varphi}}$ is a constant depending only on $\overline{\varphi}$.

Proof. Since $\frac{\overline{\varphi}(x)}{x}$ is decreasing to zero if $x\to\infty$, therefore using Lemma 3.4 and taking $\rho(x)=\overline{\varphi}(x),\ b_n=n\cdot a_n,\ \lambda_n=1$, we get

$$\sum_{n=1}^{\infty}\overline{\varphi}(b_n)\le K_{\overline{\varphi}}\cdot\sum_{n=1}^{\infty}\overline{\varphi}\left(\sum_{k=n}^{\infty}\frac{b_k}{k}\right),$$

whence the statement of Lemma 3.5 is obtained. □

4. PROOF

Proof of Theorem 2.1. Let $\varphi \in \Delta^{(2)}$ and $b_n = \varphi(|\Delta\, a_n|)$. Using Lemma 3.5 and that $\{a_n\} \in \mathcal{F}_\varphi$ we have that

$$\sum_{n=1}^{\infty} |\Delta a_n| \le K \cdot \sum_{n=1}^{\infty} \overline{\varphi}\left(\frac{1}{n}\sum_{k=n}^{\infty} \varphi(|\Delta a_n|)\right) < \infty, \tag{4.1}$$

where K depends only on φ. It now follows that the sequence $\{a_n\}$ is of bounded variation.

Further on, we use the following notations:

$$\Delta_s^{(\varphi)} := \overline{\varphi}\left\{\frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} \varphi(|\Delta\, a_k|)\right\}, \quad p' > 1$$

denotes a number for which $\frac{\varphi(x)}{x^{p'}} \uparrow$ on $(0; A)$, where $A = \sup\limits_k |\Delta a_k|$, and by q we denote a number satisfying $\frac{\varphi(x)}{x^q} \downarrow$ (see the definition of $\varphi \in \Delta^{(2)}$).

Now we will prove that for any $1 < p < p'$,

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} \le 2^{q/p} \sum_{s=1}^{\infty} 2^s \Delta_s^{(\varphi)} \tag{4.2}$$

holds.

Since for $\psi(t) = \varphi(t^{1/p})$ the function $\frac{\psi(t)}{t}$ is increasing thus using Lemma 3.3 and that $\varphi(2^{1/p}x) \le 2^{\frac{q}{p}}\varphi(x)$ we get

$$\psi\left(\frac{\sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p}{2^{s-1}}\right) \le 2^{\frac{q}{p}} \frac{\sum_{k=2^{s-1}+1}^{2^s} \varphi(|\Delta a_k|)}{2^{s-1}}. \tag{4.3}$$

From (4.3), taking into account that $\overline{\varphi}(cx) \le c\overline{\varphi}(x)$ if $c > 1$, we obtain

$$\overline{\varphi}\left[\psi\left(\frac{\sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p}{2^{s-1}}\right)\right] \le 2^{\frac{q}{p}}\overline{\varphi}\left(\frac{\sum_{k=2^{s-1}+1}^{2^s} \varphi(|\Delta a_k|)}{2^{s-1}}\right). \tag{4.4}$$

Since $\overline{\varphi}(\psi(t)) = t^{1/p}$ so from (4.4) we have

$$\Delta_s^{(p)} = \left\{\frac{\sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p}{2^{s-1}}\right\}^{\frac{1}{p}} \le 2^{q/p}\overline{\varphi}\left(\frac{\sum_{k=2^{s-1}+1}^{2^s} \varphi(|\Delta a_k|)}{2^{s-1}}\right) = 2^{q/p}\Delta_s^{(\varphi)}. \tag{4.5}$$

From (4.5), (4.2) immediately follows.

Now we show that

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(\varphi)} \le 4 \cdot \sum_{s=1}^{\infty} \overline{\varphi}\left(\frac{\sum_{k=s}^{\infty} \varphi(|\Delta a_k|)}{s}\right). \tag{4.6}$$

Since the sequence $U_s = \frac{1}{s}\sum_{k=s}^{\infty} \varphi(|\Delta a_k|)$ is monotone decreasing, we obtain

$$\begin{aligned}\sum_{s=1}^{n} 2^s \Delta_s^{(\varphi)} &= 2 \cdot \sum_{s=1}^{n} 2^{s-1}\overline{\varphi}\left(\frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} \varphi(|\Delta a_k|)\right) \\ &\le 4 \cdot \sum_{s=1}^{2^{n-1}} \overline{\varphi}(U_s) = 4 \cdot \sum_{s=1}^{2^{n-1}} \overline{\varphi}\left(\frac{1}{s}\sum_{k=s}^{\infty} \varphi(|\Delta a_k|)\right). \end{aligned} \tag{4.7}$$

Setting $n \to \infty$ we have from (4.7) the inequality (4.6).

Collecting (4.1), (4.2), (4.6), using Lemma 3.1 and Lemma 3.2, we get that (1.1) is a Fourier series and (2.1) is true. Thus Theorem 2.1 is proved. □

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