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CARLESON MEASURES FOR ANALYTIC BESOV SPACES: THE UPPER TRIANGLE CASE

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ABSTRACT. For a large family of weights ρ in the unit disc and for fixed $1 < q < p < \infty$, we give a characterization of those measures μ such that, for all functions f holomorphic in the unit disc,

$$\|f\|_{L^q(\mu)} \le C(\mu) \left(\int_{\mathbb{D}} |(1-|z|^2)f'(z)|^p \rho(z) \frac{m(dz)}{(1-|z|^2)^2} + |f(0)|^p \right)^{\frac{1}{p}}.$$

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1. INTRODUCTION

Given indices p, q, $1 < p, q < \infty$, and given a positive weight ρ on the unit disc, $\mathbb{D}$, a positive measure μ on is a *Carleson measure for* $(B_p(\rho), q)$ if the following Sobolev-type inequality holds whenever f is a function which is holomorphic in $\mathbb{D}$,

$$\|f\|_{L^q(\mu)} \le C(\mu) \left(\int_{\mathbb{D}} |(1-|z|^2)f'(z)|^p \rho(z) \frac{m(dz)}{(1-|z|^2)^2} + |f(0)|^p \right)^{\frac{1}{p}}. \tag{1.1}$$

Throughout the paper, m denotes the Lebesgue measure. If ρ is positive throughout $\mathbb{D}$ and, say, continuous, the right-hand side of (1.1) defines a norm for a Banach space of analytic functions, the *analytic Besov space* $B_p(\rho)$.

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The measure $\frac{m(dz)}{(1-|z|^2)^2}$ and the differential operator $|(1-|z|^2)f'(z)|$ should be read, respectively, as the volume element and the gradient's modulus with respect to the hyperbolic metric in $\mathbb{D}$,

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}.$$

In [2], a characterization of the Carleson measures for $(B_p(\rho), q)$ was given, when $p \le q$ and ρ is a *p-admissible weight*, to be defined below. Loosely speaking, a weight ρ is p-admissible if one can naturally identify the dual space of $B_p(\rho)$ with $B_{p'}(\rho^{1-p'})$. The weights of the form $(1-|z|^2)^s$, $s \in \mathbb{R}$, are p-admissible if and only if $-1 < s < p-1$. (Here and throughout $p^{-1} + p'^{-1} = q^{-1} + q'^{-1} = 1$).

Theorem 1.1 ([2]). *Suppose that $1 < p \le q < \infty$ and that ρ is a p-admissible weight. A positive Borel measure μ on $\mathbb{D}$ is Carleson for $(B_p(\rho), q)$ if, and only if, there is a $C_1(\mu) > 0$, so that for all $a \in \mathbb{D}$*

$$\left(\int_{S(a)} \rho(z)^{-p'/p} \left(\mu\left(S(z) \cap S(a)\right)\right)^{p'} m_h(dz)\right)^{\frac{q'}{p'}} \le C_1(\mu)\mu\left(S(a)\right). \tag{1.2}$$

For $a \in \mathbb{D}$,

$$S(a) = \left\{z \in \mathbb{D}:\ 1-|z| \le 2(1-|a|),\ \left|\frac{\arg(a\bar{z})}{2\pi}\right| \le 1-|a|\right\}$$

is the *Carleson box* with center a. The proof was based on a discretization procedure and on the solution of a two-weight inequality for the "Hardy operator on trees". Actually, when $q > p$, Theorem 1.1 holds with a "single box" condition which is simpler than (1.2). For different characterizations of the Carleson measures for analytic Besov spaces in different generality, see [13], [8], [14], [17], [18]. A short survey of results and problems is contained in [1].

In this note, we consider the Carleson measures for $(B_p(\rho), q)$ in the case $1 < q < p < \infty$. The new tool is a method allowing work in this "upper triangle" case, developed by C. Cascante, J.M. Ortega and I.E. Verbitsky in [5]. Before we state the main theorem, we introduce some notation.

For $a \in \mathbb{D}$, let $P(a) = [0, a] \in \mathbb{D}$, the segment with endpoints 0 and a. Let $1 < p < \infty$ and let ρ be a positive weight on $\mathbb{D}$. Given a positive, Borel measure μ on $\mathbb{D}$, we define its *boundary Wolff potential*, $W_{co}(\mu) = W_{co}(\rho, p; \mu)$ to be

$$W_{co}(\mu)(a) = \int_{P(a)} \rho(w)^{p'-1}\mu(S(w))^{p'-1}\frac{|dw|}{1-|w|^2}.$$

The main result of this note is the theorem below. Its statement certainly does not come as a surprise to the experts.

Theorem 1.2. *Let $1 < q < p < \infty$ and let ρ be a p-admissible weight. A positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for $(B_p(\rho), q)$ if and only if*

$$\int_{\mathbb{D}} \left(W_{co}(\mu)(z)\right)^{\frac{q(p-1)}{p-q}} \mu(dz) < \infty. \tag{1.3}$$

We say that a weight ρ is p-admissible if the following two conditions are satisfied:

(i) ρ is *regular*, i.e., there exist $\epsilon > 0$, $C > 0$ such that $\rho(z_1) \le C\rho(z_2)$ whenever z_1 and z_2 are within hyperbolic distance ϵ. Equivalently, there are $\delta < 1$, $C' > 0$ so that $\rho(z_1) \le C'\rho(z_2)$ whenever

$$\left|\frac{z_1 - z_2}{1 - z_1 z_2}\right| \le \delta < 1.$$

(ii) the weight $\rho_p(z) = (1 - |z|^2)^{p-2}\rho(z)$ satisfies the *Bekollé-Bonami $\mathcal{B}_p$ condition* ([4], [3]): There is a $C(\rho, p)$ so that for all $a \in \mathbb{D}$

$$\left(\int\limits_{S(a)} \rho_p(z) m(dz) \right) \left(\int\limits_{S(a)} \rho_p(z)^{1-p'} m(dz) \right)^{\frac{1}{p'-1}} \leq C(\rho, p) m(S(a))^p. \tag{1.4}$$

Inequalities like (1.1) have been extensively studied in the setting of Sobolev spaces. For instance, given $1 < p, q < \infty$, consider the problem of characterizing the *Maz'ya measures* for (p, q); that is, the class of the positive Borel measures μ on $\mathbb{R}$ such that the Poincarè inequality

$$\left(\int_{\mathbb{R}^n} |u|^q d\mu \right)^{\frac{1}{q}} \leq C(\mu) \left(\int_{\mathbb{R}^n} |\nabla u|^p dm \right)^{\frac{1}{p}} \tag{1.5}$$

holds for all functions u in $C_0^\infty(\mathbb{R}^n)$, with a constant independent of u. Here, we only consider the case when $1 < q < p < \infty$, and refer the reader to [16] for a comprehensive survey of these "trace inequalities". Maz'ya [11], and then Maz'ya and Netrusov [12], gave a characterization of such measures that involves suitable capacities. Later, Verbitsky [15], gave a first noncapacitary characterization.

The following noncapacitary characterization of the Maz'ya measures for $q < p$ is in [5]. For $0 < \alpha < n$, let $I_\alpha(x) = c(n, \alpha)|x|^{\alpha - n}$ be the Riesz kernel in $\mathbb{R}^n$. Recall that, for $1 < p < \infty$, (1.5) is equivalent, for $\alpha = 1$, to the inequality

$$\left(\int_{\mathbb{R}^n} |I_\alpha \star v|^q d\mu \right)^{\frac{1}{q}} \leq C(\mu) \left(\int_{\mathbb{R}^n} |v|^p dm \right)^{\frac{1}{p}} \tag{1.6}$$

with a constant $C(\mu)$, independent of $v \in L^p(\mathbb{R}^n)$.

Now, let $B(x, r)$ denote the ball in $\mathbb{R}^n$, having its center at x and radius r. The *Hedberg-Wolff* potential $W_{\alpha,p}$ of μ is

$$W_{\alpha,p}(\mu)(x) = \int_0^\infty \left(\frac{\mu(B(x, r))}{r^{n-\alpha p}} \right)^{p'-1} \frac{dr}{r}.$$

Theorem 1.3 ([5]). *If $1 < q < p < \infty$ and $0 < \alpha < n$, μ satisfies (1.6) if and only if*

$$\int_{\mathbb{R}^n} (W_{\alpha,p}(\mu))^{\frac{q(p-1)}{p-q}} d\mu < \infty. \tag{1.7}$$

Comparing the different characterizations for the analytic-Besov and the Sobolev case, we see at work the heuristic principle according to which the relevant objects for the analysis in Sobolev spaces (e.g., Euclidean balls, or the potential $W_{\alpha,p}$) have as holomorphic counterparts similar objects, who live *near the boundary* (e.g., Carleson boxes, or the potential W_{co}). This is expected, since a holomorphic function cannot behave badly inside its domain. Another simple, but important, heuristic principle is that *holomorphic functions are essentially discrete*. By this, we mean that, for many purposes, we can consider a holomorphic function in the unit disc as if it were constant on discs having radius comparable to their distance to the boundary. (For positive harmonic functions, this is just Harnack's inequality). Based on these considerations, one might think that the problem of characterizing the Carleson measures for $(B_p(\rho), q)$ might be reduced to some discrete problem. This is in fact true, and it is the main tool in the proof of Theorem 1.2.

The idea, already exploited in [2], is to consider $(1 - |z|^2)f'(z)$ constants on sets that form a Whitney decomposition of $\mathbb{D}$. The Whitney decomposition has a natural tree structure, hence, the Carleson measure problem leads to a weighted inequality on trees.

The discrete result is the following. Let T be a tree, i.e., a connected, loopless graph, that we do not assume to be locally finite; see Section 3 for complete definitions and notation. Let $o \in T$ be a fixed vertex, the *root* of T. There is a partial order on T defined by: $x \le y,\ x, y \in T$, if $x \in [o, y]$, the geodesic joining o and y. Let $\varphi\colon T \to \mathbb{C}$. We define $\mathcal{I}\varphi$, the *Hardy operator on T, with respect to o,* applied to φ, by

$$\mathcal{I}\varphi(x) = \sum_{o}^{x} \varphi(y) = \sum_{y \in [o,x]} \varphi(y).$$

A *weight* ρ on T is a positive function on T.

For $x \in T$, let $S(x) = \{y \in T : y \ge x\}$. $S(x)$ is the *Carleson box with vertex x* or the *successors' set* of x. Also, let $P(x) = \{z \in T \colon o \le z \le x\}$ be the *predecessors' set* of x. Given a positive weight ρ and a nonnegative function μ on T, and given $1 < p < \infty$, define $W(\mu) = W(\rho, p; \mu)$, the *discrete Wolff potential* of μ,

$$W(\mu)(x) = \sum_{y \in P(x)} \rho(y)^{1-p'} \mu(S(y))^{p'-1}. \tag{1.8}$$

Theorem 1.4. *Let $1 < q < p < \infty$ and let ρ be a weight on T. For a nonnegative function μ on T, the following are equivalent :*

(1) *For some constant $C(\mu) > 0$ and all functions φ*

$$\left(\sum_{x \in T} |\mathcal{I}\varphi(x)|^q \mu(x)\right)^{\frac{1}{q}} \le C(\mu) \left(\sum_{x \in T} |\varphi(x)|^p \rho(x)\right)^{\frac{1}{p}}. \tag{1.9}$$

(2) *We have the inequality*

$$\sum_{x \in T} \mu(x) \left(W(\mu)(x)\right)^{\frac{q(p-1)}{p-q}} < \infty. \tag{1.10}$$

A different characterization of the measures μ for which (1.9) holds is given in [6] Theorem 3.3, in the more general context of *thick trees* (i.e., trees in which the edges are copies of intervals of the real line). The necessary and sufficient condition given in [6], however, seems more difficult to verify than (1.10), at least in our simple context.

The paper is structured as follows. In Section 2, we show that the problem of characterizing the measures μ for which (1.1) holds is completely equivalent, if ρ is p-admissible, to the corresponding problem for the Hardy operator on trees, $\mathcal{I}$. The idea, already present in [2], is to replace the unit disc by one of its Whitney decompositions, endowed with its natural tree structure, and the integral along segments by the sum along tree-geodesics. In Section 3, following [5], Theorem 1.4 is proved, and the problem on trees is solved. Section 2 and Section 3 together, show that (1.1) is equivalent to a condition which is the discrete analogue of (1.3). Unfortunately, this discrete condition depends on the chosen Whitney decomposition. In Section 4, we show that the discrete condition is in fact equivalent to (1.3). In the course of the proof, we will see that the Carleson measure problem in the unit disc is equivalent to a number of its different discrete "metaphors" on a suitable graph.

Actually, this route to the proof of Theorem 1.2 is not the shortest possible. In fact, we could have carried the discretization directly over a graph, skipping the repetition of some arguments. We chose to do otherwise for two reasons. First, the tree situation is slightly easier to handle, and it leads, already in Section 2, to a characterization of the Carleson measures for $(B_p(\rho), q)$. Second, it can be more easily compared with the proof of the characterization theorem for the case $q \ge p$, which was obtained in [2] working on trees.

It should be mentioned that [5] has some results for spaces of holomorphic functions, which are different from those considered in this article.

2. DISCRETIZATION

In this section, Theorem 2.5, we show that, if ρ is p-admissible, then the problem of characterizing the Carleson measures for $B_p(\rho)$ is equivalent to a two-weight inequality on trees. This fact is already implicit in [2], but, here, our formulation stresses more clearly the interplay between the discrete and the continuous situation. In the context of the weighted Bergman spaces, a similar approach to the Carleson measures problem was employed by Luecking [9], who also obtained a characterization theorem in the upper triangle case [10].

First, we recall some facts on Bergman and analytic Besov spaces.

Let $1 < p < \infty$ be fixed and let ρ be a weight on $\mathbb{D}$. The *Bergman space* $A_p(\rho)$ is the space of those functions f that are holomorphic in $\mathbb{D}$ and such that

$$\|f\|_{A_p(\rho)}^p = \int_{\mathbb{D}} |f(z)|^p \rho(z) m(dz)$$

is finite. Define, for $f,\ g \in A_2 \equiv A_2(1)$,

$$\langle f, g\rangle_{A_2} = \int_{\mathbb{D}} f(z)\overline{g(z)} m(dz).$$

Let $A_p(\rho)^*$ be the dual space of $A_p(\rho)$. We identify $g \in A_{p'}(\rho^{1-p'})$ with the functional on $A_p(\rho)$

$$\Lambda_g \colon f \mapsto \langle f, g\rangle_{A_2}. \tag{2.1}$$

By Hölder's inequality we have that $A_p^*(\rho) \subseteq A_{p'}(\rho^{1-p'})$. Condition (1.4) shows that the reverse inclusion holds.

Theorem 2.1 (Bekollé-Bonami [4], [3])**.** *If the weight ρ satisfies (1.4) then $g \mapsto \Lambda_g$, where Λ_g defined in (2.1) is an isomorphism of $A_{p'}(\rho^{1-p'})$ onto $A_p^*(\rho)$.*

We need some consequences of Theorem 2.1, whose proof can be found in [2], §2 and §4.

Let F, G be holomorphic functions in $\mathbb{D}$,

$$F(z) = \sum_0^\infty a_n z^n, \qquad G(z) = \sum_0^\infty b_n z_n.$$

Define

$$\langle F, G\rangle_{\mathcal{D}^*} = \sum_1^\infty n a_n \overline{b_n} = \int_{\mathbb{D}} F'(z)\overline{G'(z)} m(dz)$$

and

$$\langle F, G\rangle_{\mathcal{D}} = a_0\overline{b_0} + \sum_1^\infty n a_n \overline{b_n} = F(0)\overline{G(0)} + \langle F, G\rangle_{\mathcal{D}^*}.$$

Lemma 2.2. *Let ρ be a weight satisfying (1.4). Then $B_{p'}(\rho^{1-p'})$ is the dual of $B_p(\rho)$ under the pairing $\langle\cdot,\cdot\rangle_{\mathcal{D}}$. i.e., each functional Λ on $B_p(\rho)$ can be represented as*

$$\Lambda f = \langle f, g\rangle_{\mathcal{D}}, \qquad f \in B_p(\rho)$$

for a unique $g \in B_{p'}(\rho^{1-p'})$.

The reproducing kernel of $\mathcal{D}$ with respect to the product $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ is

$$\phi_z(w) = 1 + \log\frac{1}{1-w\bar{z}}$$

i.e., if $f \in \mathcal{D}$, then

$$f(z) = \langle f, \phi_z\rangle_{\mathcal{D}} = \int_{\mathbb{D}} f'(w)\overline{\left(1+\log\frac{1}{1-\bar{z}w}\right)'}m(dw) + f(0).$$

Lemma 2.3. *Let ρ be an admissible weight, $1<p<\infty$. Then, ϕ_z is a reproducing kernel for $B_{p'}(\rho^{1-p'})$. i.e., if $G \in B_{p'}(\rho^{1-p'})$, then*

$$G(z) = \langle G, \phi_z\rangle_{\mathcal{D}}. \tag{2.2}$$

In particular, point evaluation is bounded on $B_{p'}(\rho^{1-p'})$.

Observe that (1.4) is symmetric in (ρ, p) and $(\rho^{1-p'}, p')$ and hence the same conclusion holds for $B_p(\rho)$.

Now, let μ be a positive bounded measure on $\mathbb{D}$ and define

$$\langle F, G\rangle_\mu = \langle F, G\rangle_{L^2(\mu)} = \int_{\mathbb{D}} F(z)\overline{G(z)}\mu(dz).$$

μ is Carleson for $(B_p(\rho), p, q)$ if and only if

$$Id : B_p(\rho) \to L^q(\mu)$$

is bounded. In turn, this is equivalent to the boundedness, with the same norm, of its adjoint $\Theta = Id^*$,

$$\Theta : L^{q'}(\mu) \to (B_p(\rho))^* \equiv B_{p'}(\rho^{1-p'}),$$

where we have used the duality pairings $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ and $\langle\cdot,\cdot\rangle_\mu$, and Lemma 2.2.

By Lemma 2.3,

$$\begin{aligned}\Theta G(z) &= \langle \Theta G, \phi_z\rangle_{\mathcal{D}} = \langle G, \phi_z\rangle_{L^2(\mu)}\\ &= \int_{\mathbb{D}}\left(1+\log\frac{1}{1-z\bar{w}}\right)G(w)\mu(dw).\end{aligned}$$

For future reference, we state this as

Lemma 2.4. *If ρ is a p-admissible weight, the adjoint of $Id : B_p(\rho) \to L^q(\mu)$ is the operator*

$$\Theta : L^{q'}(\mu) \to (B_p(\rho))^* \equiv B_{p'}(\rho^{1-p'})$$

defined by

$$\Theta G(z) = \int_{\mathbb{D}}\left(1+\log\frac{1}{1-z\bar{w}}\right)G(w)\mu(dw). \tag{2.3}$$

Consider, now, a dyadic Whitney decomposition of $\mathbb{D}$. Namely, for integer $n \ge 0,\ 1 \le m \le 2^n$, let

$$\Delta_{n,m} = \left\{z \in \mathbb{D}\colon 2^{-n-1} \le 1-|z| \le 2^{-n},\ \left|\frac{\arg(z)}{2\pi} - \frac{m}{2^n}\right| \le 2^{-(n+1)}\right\}.$$

These boxes are best seen in polar coordinates. It is natural to consider the Whitney squares as indexed by the vertices of a dyadic tree, T_2. Thus the vertices of T_2 are

$$\{\alpha | \alpha = (n,m),\ n \ge 0 \text{ and } 1 \le m \le 2^n,\ m,n \in \mathbb{N}\} \tag{2.4}$$

and we say that there is an edge between $(n,m),\ (n',m')$ if $\Delta_{(n,m)}$ and $\Delta_{(n',m')}$ share an arc of a circle. The root of T_2 is, by definition, $(0,1)$. Here and throughout we will abuse notation and, when convenient, identify the vertices of such a tree with the sets for which they are indices. Here we identify α and Δ_α. Thus, there are four edges having $(0,1)$ as an endpoint, each other box being the endpoint of exactly three edges.

Given a positive, regular weight ρ on $\mathbb{D}$, we define a weight on T_2, still denoted by ρ. If $\alpha \in T_2$ and if z_α be, say, the center of the box $\alpha \subset \mathbb{D}$, then, $\rho(\alpha) = \rho(z_\alpha)$. By the regularity assumption, the choice of z_α does not matter in the estimates that follow.

Theorem 2.5. *Let $1 < q,\ p < \infty$ and let ρ be a p-admissible weight. A positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for $(B_p(\rho), q)$ if and only if the following inequality holds, with a constant C which is independent of $\varphi\colon T_2 \to \mathbb{R}$.*

$$\left(\sum_{x\in T_2} |\mathcal{I}\varphi(x)|^q \mu(x)\right)^{\frac{1}{q}} \leq C \left(\sum_{y\in T_2} |\varphi(y)|^p \rho(y)\right)^{\frac{1}{p}}. \tag{2.5}$$

Proof. It is proved in [2] (§4, Theorem 12, proof of the sufficiency condition) that (2.5) is sufficient for μ to be a Carleson measure.

We come, now, to necessity. Without loss of generality, assume that supp$(\mu) \subseteq \{z\colon |z| \leq 1/2\}$. By the remarks preceding the proof, Lemma 2.2 and Lemma 2.3, IF μ is Carleson, then Θ is bounded from $L^{q'}(\mu)$ to $B_{p'}(\rho^{1-p'})$. Consider, now, functions $g \in L^{q'}(\mu)$, having the form

$$g(w) = \frac{|w|}{\overline{w}} h(w),$$

where $h \geq 0$ and h is constant on each box $\alpha \in T_2$, $h|_\alpha = h(\alpha)$. The boundedness of Θ implies

$$\begin{aligned}
C\left(\sum_{\alpha\in T_2} |h(\alpha)|^{q'} \mu(\alpha)\right)^{\frac{1}{q'}} &= \left(\int_{\mathbb{D}} |g|^{q'} d\mu\right)^{\frac{1}{q'}} \\
&\geq \|\Theta g\|_{B_{p'}(\rho^{1-p'})} \\
&\geq \left(\int_{\mathbb{D}} \left|\int_{\mathbb{D}} \frac{1-|z|^2}{1-z\overline{w}} |w| h(w)\mu(dw)\right|^{p'} \rho(z)^{1-p'} \frac{m(dz)}{(1-|z|^2)^2}\right)^{\frac{1}{p'}}.
\end{aligned}$$

For $z \in \mathbb{D}$, let $\alpha(z) \in T_2$ be the Whitney box containing z. By elementary estimates,

$$\operatorname{Re}\left(\frac{|w|(1-|z|^2)}{1-z\overline{w}}\right) \geq 0 \tag{2.6}$$

if $w \in \mathbb{D}$, and

$$\operatorname{Re}\left(\frac{|w|(1-|z|^2)}{1-z\overline{w}}\right) \geq c > 0, \text{ if } w \in S(\alpha(z))$$

for some universal constant c. If $\alpha(z) = o$ is the root of T_2, the latter estimate holds, say, only on one half of the box o, and this suffices for the calculations below.

Using this, and the fact that all our Whitney boxes have comparable hyperbolic measure, we can continue the chain of inequalities

$$\geq \left(\int_{\mathbb{D}} \left| \int_{\mathbb{S}(\alpha(z))} \frac{1-|z|^2}{1-z\overline{w}} |w| h(w) \mu(dw) \right|^{p'} \rho^{1-p'}(z) \frac{m(dz)}{(1-|z|^2)^2} \right)^{\frac{1}{p'}}$$

$$\geq c \left(\int_{\mathbb{D}} \left(\sum_{\beta \in S(\alpha(z))} h(\beta)\mu(\beta) \right)^{p'} \rho^{1-p'}(z) \frac{m(dz)}{(1-|z|^2)^2} \right)^{\frac{1}{p'}}$$

$$\geq c \left(\sum_{\alpha} \left(\sum_{\beta \in S(\alpha)} h(\beta)\mu(\beta) \right)^{p'} \rho(\alpha)^{1-p'} \right)^{\frac{1}{p'}}.$$

Let $\mathcal{I}^*$, defined on functions $\varphi\colon T_2 \to \mathbb{R}$, be the operator

$$\mathcal{I}^*\varphi(\alpha) = \sum_{\beta \in S(\alpha)} \varphi(\beta)\mu(\beta).$$

It is readily verified that $\mathcal{I}^*$ is the adjoint of $\mathcal{I}$, in the sense that

$$\sum_{T_2} \mathcal{I}^*\psi(\alpha)\varphi(\alpha) = \sum_{T_2} \psi(\alpha)\mathcal{I}\varphi(\alpha)\mu(\alpha).$$

Then, the chain of inequalities above shows that

$$\mathcal{I}^*\colon L^{q'}(\mu) \to L^{p'}(\rho^{1-p'})$$

is a bounded operator. In turn, this is equivalent to the boundedness of

$$\mathcal{I}\colon L^p(\rho) \to L^q(\mu).$$

□

3. A Two-weight Hardy Inequality on Trees

In this section we prove Theorem 1.4.

Let T be a tree. We use the same name T for the tree and for its set of vertices. We do not assume that T is locally finite; a vertex of T can be the endpoint of infinitely many edges. If $x, y \in T$, the *geodesic between* x *and* y, $[x, y]$, is the set $\{x_0, \dots, x_n\}$, where $x_0 = x,\ x_n = y,\ x_{j-1}$ is adjacent to x_j (i.e., x_{j-1} and x_j are endpoints of an edge), and the vertices in $[x, y]$ are all distinct. We let $[x, x] = \{x\}$. If x, y are as above, we let $d(x, y) = n$. Let $o \in T$ be a fixed *root*. We say that $x \leq y$, $x, y \in T$, if $x \in [o, y]$. $\leq$ is a partial order on T. For $x \in T$, the *Carleson box* of vertex x (or the *set of successors of* x) is $S(x) = \{y \in T\colon y \geq x\}$. We will sometimes write $[o, x] = P(x)$, the *set of predecessors of* x.

Theorem 3.1. *Let* $1 < q < p < \infty$ *and let* ρ *be a weight on* T. *For a nonnegative function* μ *on* T*, the following are equivalent:*

(1) *For some constant* $C(\mu) > 0$ *and all functions* φ

$$\left(\sum_{x \in T} |\mathcal{I}\varphi(x)|^q \mu(x) \right)^{\frac{1}{q}} \leq C(\mu) \left(\sum_{x \in T} |\varphi(x)|^p \rho(x) \right)^{\frac{1}{p}}. \tag{3.1}$$

(2) *We have the inequality*

$$\sum_{x\in T}\mu(x)\left(W(\mu)(x)\right)^{\frac{q(p-1)}{p-q}}<\infty. \tag{3.2}$$

As a consequence of Theorems 3.1 and 2.5, we obtain a characterization of the Carleson measures for $(B_p(\rho),q)$.

Corollary 3.2. *A measure μ in the unit disc is Carleson for $(B_p(\rho),q)$ if, and only if,*

$$\sum_{\alpha\in T_2}\mu(\alpha)\left(W(\mu)(\alpha)\right)^{\frac{q(p-1)}{p-q}}<\infty. \tag{3.3}$$

Proof. First, we show that (3.2) implies (3.1). By duality, it suffices to show that, if (3.2) holds, then $\mathcal{I}^*$,

$$\mathcal{I}^*\varphi(x)=\sum_{y\in S(x)}\varphi(y)\mu(y)$$

is a bounded map from $L^{q'}(\mu)$ to $L^{p'}(\rho^{1-p'})$. Without loss of generality, we can test $\mathcal{I}^*$ on positive functions. Let $g\geq 0$. Then

$$\begin{aligned}\|\mathcal{I}^*g\|^{p'}_{L^{p'}(\rho^{1-p'})}&=\sum_{x\in T}\rho(x)^{1-p'}\left(\sum_{y\in S(x)}g(y)\mu(y)\right)^{p'}\\ \text{by definition of } W,\ &=\sum_{y\in T}g(y)\mu(y)W(g\mu)(y).\end{aligned}$$

Define, now, the maximal function

$$M_\mu g(y)=\max_{z\in P(y)}\frac{\sum_{t\in S(z)}g(t)\mu(t)}{\mu(S(z))}. \tag{3.4}$$

The following lemma will be proved at the end of the proof of Theorem 3.1. It can be considered as a discrete, boundary version of the weighted maximal theorem of R. Fefferman [7].

Lemma 3.3. *If $1<s<\infty$ and μ is a bounded measure on T, then M_μ is bounded on $L^s(\mu)$.*

Since $g\geq 0$,

$$\begin{aligned}W(g\mu)(y)&\leq\sum_{P(y)}\rho(x)^{1-p'}\mu(S(x))^{p'-1}\left(M_\mu g(y)\right)^{p'-1}\\ &=W(\mu)(y)\left(M_\mu g(y)\right)^{p'-1}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{I}^*g\|^{p'}_{L^{p'}(\rho^{1-p'})}&\leq\sum_{y\in T}g(y)\mu(y)W(\mu)(y)\left(M_\mu g(y)\right)^{p'-1}\\ &\leq\left(\sum_{y\in T}M_\mu g(y)^{(p'-1)r}\mu(y)\right)^{\frac{1}{r}}\left(\sum_{y\in T}g(y)^{r'}(W(\mu)(y))^{r'}\mu(y)\right)^{\frac{1}{r'}}\end{aligned}$$

by Hölder's inequality, with $r=\frac{q'}{p'-1}>1$,

$$\leq C\|g\|^{p'-1}_{L^{q'}(\mu)}\left(\sum_{y\in T}g(y)^{\lambda r'}\mu(y)\right)^{\frac{1}{\lambda r'}}\left(\sum_{y\in T}(W(\mu)(y))^{\lambda' r'}\mu(y)\right)^{\frac{1}{\lambda' r'}}$$

by Lemma 3.3 and Hölder's inequality, with $\lambda = q' - p' + 1 > 1$,

$$= C\|g\|_{L^{q'}(\mu)}^{p'} \left(\sum_{y\in T} (W(\mu)(y))^{\frac{(p-1)q}{p-q}} \mu(y) \right)^{\frac{p-q}{(p-1)q}}.$$

This proves one implication.

We show that, conversely, (3.1) implies (3.2). By hypothesis and duality, we have, for $g \geq 0$,

$$\begin{aligned} \|g\|_{L^{q'}(\mu)}^{p'} &\geq C \sum_{y\in T} \mathcal{I}^* g(y)^{p'} \rho(y)^{1-p'} \\ &= \sum_{y\in T} \rho(y)^{1-p'} \mu(S(y))^{p'} \left(\frac{\sum_{x\in S(y)} g(x)\mu(x)}{\mu(S(y))} \right)^{p'}. \end{aligned}$$

Replace $g = (M_\mu h)^{\frac{1}{p'}}$, with $h \geq 0$. By Lemma 3.3, since $q' > p'$,

$$\begin{aligned} \|h\|_{L^{\frac{q'}{p'}}(\mu)} &\geq C \|M_\mu h\|_{L^{\frac{q'}{p'}}(\mu)} \\ &= C \| \left(M_\mu h\right)^{1/p'} \|_{L^{q'}}^{p'}(\mu) \\ &\geq C \sum_{y\in T} e(y) \left(\frac{\sum_{x\in S(y)} (M_\mu h(x))^{\frac{1}{p'}} \mu(x)}{\mu(S(y))} \right)^{p'}, \end{aligned}$$

where $e(y) = \rho(y)^{1-p'} \mu(S(y))^{p'}$,

$$\begin{aligned} &\geq C \sum_{y\in T} e(y) \left(\frac{\sum_{x\in S(y)} \left(\sum_{t\in S(y)} h(t)\mu(t) \big/ \mu(S(y)) \right)^{\frac{1}{p'}} \mu(x)}{\mu(S(y))} \right)^{p'} \\ &\geq C \sum_{y\in T} \frac{e(y)}{\mu(S(y))} \left(\sum_{t\in S(y)} h(t)\mu(t) \right) \\ &= C \sum_{t\in T} \mu(t)h(t) \left(\sum_{y\in T} \frac{e(y)}{\mu(S(y))} \chi_{S(y)} \right)(t). \end{aligned}$$

By duality, then, we have that

$$\sum_{y\in T} \frac{e(y)}{\mu(S(y))} \chi_{S(y)} \in L^{(q'/p')'}(\mu) = L^{\frac{q(p-1)}{p-q}}(\mu).$$

Hence,

$$\begin{aligned} \infty &> \sum_{x\in T} \left(\sum_{y\in T} \frac{e(y)}{\mu(S(y))} \chi_{S(y)}(x) \right)^{\frac{q(p-1)}{p-q}} \mu(x) \\ &= \sum_{x\in T} \mu(x) \left(W(\mu)(x)\right)^{\frac{q(p-1)}{p-q}}, \end{aligned}$$

which is the desired conclusion. □

As a final remark, let us observe that the potential W admits the following, suggestive formulation:

$$W(\mu) = \mathcal{I}\left(\rho^{1-p'}\left(\mathcal{I}^* Id\right)^{p'-1}\right),$$

where Id is the identity operator.

Condition (1.10) might, then, be reformulated as

$$\mathcal{I}\left(\rho^{1-p'}\left(\mathcal{I}^* Id\right)^{p'-1}\right) \in L^{\frac{q(p-1)}{p-q}}(\mu).$$

Proof of Lemma 3.3. The argument is a caricature of the classical one. By intepolation, it suffices to show that M_μ is of weak type $(1,1)$. For $f \geq 0$ on T and $t > 0$, let $E(t) = \{M_\mu > f > t\}$. If $x \in E(t)$, there exists $z = z(x) \in P(x)$, such that

$$t\mu(S(z)) < \sum_{y \in S(z)} f(y)\mu(y).$$

Let I be the set of such z's. By the tree structure, there exists a subset J of I, which is maximal, in the sense that, for each z in I, there is a w in J so that $w \geq z$. Hence,

$$E(t) \subseteq \cup_{z \in I} S(z) = \cup_{w \in J} S(w)$$

the latter union being disjoint. Thus,

$$\mu(E(t)) \leq \sum_{w \in J} \mu(S(w)) \leq \frac{1}{t}\|f\|_{L^1(\mu)},$$

which is the desired inequality. □

4. EQUIVALENCE OF TWO CONDITIONS

The last step in the proof of Theorem 1.2 consists of showing that condition (3.3) is equivalent to (1.3). In order to do so, we introduce in T_2, the set of the Whitney boxes in which $\mathbb{D}$ was partitioned, a graph structure, which is richer than the tree structure we have considered so far.

Let T_2 be the *set* defined in Section 2. We make T_2 into a graph G structure as follows. For $\alpha, \beta \in T_2$, to say that "*there is an edge of G between α and β*" is to say that the closures α and β share an arc or a straight line. For $\alpha, \beta \in G$, the distance between α and β, $d_G(\alpha, \beta)$, is the minimum number of edges in a path between α and β. The ball of center α and radius $k \in \mathbb{N}$ in G will be denoted by $B(\alpha, k) = \{\beta \in G : d_G(\alpha, \beta) \leq k\}$. In G, we maintain the partial order given by the original tree structure. In particular, we still have the tree geodesics $[0, \alpha]$.

Let $k \geq 0$ be an integer. For $\alpha \in G$, define

$$P_k(\alpha) = \{\beta \in G : d_G(\beta, [o, \alpha]) \leq k\}$$

and, dually,

$$S_k(\alpha) = \{\beta \in G : \alpha \in P_k(\beta)\}.$$

Observe that $\beta \in S_k(\alpha)$ if and only if $[0, \beta] \cap B(\alpha, k)$ is nonempty. Clearly, $P_0 = P$ and $S_0 = S$ are the sets defined in the tree case. The corresponding operators $\mathcal{I}_k$ and $\mathcal{I}_k^*$ are defined as follows. For $f : G \to \mathbb{R}$,

$$\mathcal{I}_k f(\alpha) = \sum_{\beta \in P_k(\alpha)} f(\beta) \tag{4.1}$$

and

$$\mathcal{I}_k^* f(\alpha) = \sum_{\beta \in S_k(\alpha)} f(\beta)\mu(\beta). \tag{4.2}$$

As before, $\mathcal{I}_k$ and $\mathcal{I}_k^*$ are dual to each other. That is, if $L^2(G)$ is the L^2 space on G, with respect to the counting measure,

$$\langle \mathcal{I}_k\varphi, \psi\rangle_{L^2(\mu)} = \langle \varphi, \mathcal{I}_k^*\psi\rangle_{L^2(G)}.$$

For each k, we have a discrete potential

$$W_k(\mu)(x) = \sum_{y\in P_k(x)} \rho(y)^{1-p'}\mu(S_k(y))^{p'-1}$$

and a *[COV]-condition*

$$\sum_{x\in T} \mu(x)\left(W_k(\mu)(x)\right)^{\frac{q(p-1)}{p-q}} < \infty. \tag{4.3}$$

If $f \geq 0$ on G, then $\mathcal{I}_k f \geq \mathcal{I} f$, pointwise on G. The estimates for all these operators, however, behave in the same way.

Proposition 4.1. *Let μ be a measure and ρ a p-admissible weight on $\mathbb{D}$, $1 < q < p < \infty$. Also, let μ and ρ denote the corresponding weights on G.*

Then, the following conditions are equivalent

(i) *There exists $C > 0$ such that (1.1) holds, that is*

$$\|f\|_{L^q(\mu)} \leq C(\mu)\left(\int_{\mathbb{D}} |(1-|z|^2)f'(z)|^p\rho(z)\frac{m(dz)}{(1-|z|^2)^2} + |f(0)|^p\right)^{\frac{1}{p}}$$

whenever f is holomorphic on $\mathbb{D}$.

(ii) *For $k \geq 2$, there exists $C_k > 0$ such that*

$$\left(\sum_{x\in T} |\mathcal{I}_k\varphi(x)|^q\,\mu(x)\right)^{\frac{1}{q}} \leq C_k(\mu)\left(\sum_{x\in T} |\varphi(x)|^p\,\rho(x)\right)^{\frac{1}{p}}. \tag{4.4}$$

(iii) *The following inequality holds,*

$$\sum_{x\in T} \mu(x)\left(W_k(\mu)(x)\right)^{\frac{q(p-1)}{p-q}} < \infty. \tag{4.5}$$

(iv) *(1.3) holds,*

$$\int_{\mathbb{D}} \left(W_{co}(\mu)(z)\right)^{\frac{q(p-1)}{p-q}}\mu(dz) < \infty. \tag{4.6}$$

(v) *(1.10) holds,*

$$\sum_{x\in T} \mu(x)\left(W(\mu)(x)\right)^{\frac{q(p-1)}{p-q}} < \infty. \tag{4.7}$$

(vi) *(1.9) holds for some $C > 0$,*

$$\left(\sum_{x\in T} |\mathcal{I}\varphi(x)|^q\,\mu(x)\right)^{\frac{1}{q}} \leq C(\mu)\left(\sum_{x\in T} |\varphi(x)|^p\,\rho(x)\right)^{\frac{1}{p}}. \tag{4.8}$$

Proof. We prove that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (i).

The implications (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (i) were proved in Theorems 1.4 and 1.2, respectively. (i) $\Longrightarrow$ (ii) can be proved by the same argument used in the proof of Theorem 2.5, with minor changes only. The key is the estimate (2.6).

The proof that (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) is easy. Observe that $W_k(\mu)$ increases with k, hence that (iii) with $k = n$ implies (iii) with $k = n - 1$. In particular, it implies (v), which corresponds to

$k = 0$. Let $z \in \mathbb{D}$, and let $\alpha(z) \in G$ be the box containing z. Then, it is easily checked that, if $k \geq 2$,

$$S(\alpha(z)) \subset S(z) \subset S_k(\alpha(z)).$$

To show the implication (iii) $\Longrightarrow$ (iv), observe that

$$\begin{aligned}
W_{co}(\mu)(z) &= \int_{P(z)} \rho(w)^{p'-1} \mu(S(w))^{p'-1} \frac{|dw|}{1-|w|^2} \\
&\leq C \sum_{\beta=o}^{\alpha(z)} \rho(\beta)^{p'-1} \mu(S(\beta))^{p'-1} \\
&\leq C \sum_{\beta \in P_k(\alpha(z))} \rho(\beta)^{p'-1} \mu(S_k(\beta))^{p'-1} \\
&= C W_k(\mu)(\alpha(z)),
\end{aligned}$$

hence

$$\begin{aligned}
\int_{\mathbb{D}} \left(W_{co}(\mu)(z)\right)^{\frac{q(p-1)}{p-q}} \mu(dz) &\leq C \sum_{\alpha \in G} \sup_{z \in \alpha} \left(W_{co}(\mu)(z)\right)^{\frac{q(p-1)}{p-q}} \mu(\alpha) \\
&\leq C \sum_{\alpha \in G} \left(W_k(\mu)(\alpha)\right)^{\frac{q(p-1)}{p-q}} \mu(\alpha)
\end{aligned}$$

as wished.

For $\gamma \in G$, let γ^- be the predecessor of $\gamma : \gamma^- \in [o, \gamma]$ and $d_G(\gamma, \gamma^-) = 1$. For the implication (iv) $\Longrightarrow$ (v), we have

$$\begin{aligned}
W_{co}(\mu)(z) &= \int_{P(z)} \rho(w)^{p'-1} \mu(S(w))^{p'-1} \frac{|dw|}{1-|w|^2} \\
&\geq C \sum_{\beta=o}^{\alpha(z)^-} \rho(\beta)^{p'-1} \mu(S(\beta))^{p'-1} \\
&\geq C \sum_{\beta=o}^{\alpha(z)} \rho(\beta)^{p'-1} \mu(S(\beta))^{p'-1} \\
&= C W(\mu)(\alpha(z)).
\end{aligned}$$

In the second last inequality, we used the fact that $S(\alpha) \subset S(\alpha^-)$. Then,

$$\begin{aligned}
\int_{\mathbb{D}} \left(W_{co}(\mu)(z)\right)^{\frac{q(p-1)}{p-q}} \mu(dz) &\geq C \sum_{\alpha \in G} \inf_{z \in \alpha} \left(W_{co}(\mu)(z)\right)^{\frac{q(p-1)}{p-q}} \mu(\alpha) \\
&\geq C \sum_{\alpha \in G} \left(W(\mu)(\alpha)\right)^{\frac{q(p-1)}{p-q}} \mu(\alpha)
\end{aligned}$$

and this shows that (iv) $\Longrightarrow$ (v).

We are left with the implication (ii) $\Longrightarrow$ (iii). The proof follows, line by line, that of Theorem 1.4 in Section 3. One only has to modify the definition of the maximal function

$$M_{k,\mu} g(y) = \max_{z \in P_k(y)} \frac{\sum_{t \in S_k(z)} g(t)\mu(t)}{\mu(S_k(z))}. \tag{4.9}$$

We just have to verify that $M_{k,\mu}$ is bounded on $L^s(\mu)$, if $1 < s < \infty$. It suffices to show that $M_{k,\mu}$ is of type weak $(1,1)$ and this, in turn, boils down to the covering lemma that follows.

Lemma 4.2. *There exists a constant $L > 0$ with the following property. Let F be any set in G,*

$$F \subseteq \cup_{z\in I} S_k(z)$$

where $I \subseteq G$ is an index set. Then, there exists $J \subseteq I$ such that

$$F \subseteq \cup_{z\in J} S_k(z)$$

and, for all $x \in G$,

$$\sharp\{w \in J : x \in S_k(w)\} \le L$$

where $\sharp A$ is the number of elements in the set A.

Proof of the lemma. For simplicity, we prove the lemma when $k = 2$. Incidentally, this suffices to finish the proof of Theorem 1.2.

It suffices to show that, if z_j are points in G, $j = 1, 2, 3$, and $\cap_{j=1}^3 S_2(z_j)$ is nonempty, then one of the $S_2(z_j)$'s, say $S_2(z_1)$, is contained in the union of the other two. In fact, this gives $L = 2$ in the lemma.

Let $z \in G$, $d_G(z, w) \ge 3$. Let z^{-2} be the point w in $[o, z]$ such that $d_G(o, w) = 2$ and let z^* be the only point w in G such that

$$w \in S_2(z),\ d_G(o, w) = d_G(o, z) - 1 \text{ and } w \notin S_0(z^{-2})$$

where $S_0(z) = S(z)$ is the same Carleson box introduced in Section 3. Then, one can easily see that

$$S_2(z) = S_0(z^{-2}) \cup S_0(z^*)$$

the union being disjoint.

Let now z_1, z_2, z_3 be as above, with $d_G(o, z_1) \ge d_G(o, z_2) \ge d_G(o, z_3)$. Then, $d_G(o, z_1^{-2}) \ge d_G(o, z_2^{-2}) \ge d_G(o, z_3^{-2})$ and z_2 is a point within d_G distance 1 from $S_2(z_3)$. If $S_2(z_2) \subseteq S_2(z_3)$, there is nothing to prove. Otherwise,

$$S_2(z_2) \cup S_2(z_3) = S_0(z_3^{-2}) \cup S_0(z_3^*) \cup S_0(w),$$

where $w = z_2^{-2}$ or $w = z_2^*$, respectively, the union being disjoint, and

$$S_2(z_2) \cap S_2(z_3) = S_0(\xi),$$

where $\xi = z_2^*$ or $\xi = z_2^{-2}$, respectively. In the first case, since $d_G(o, z_1) \ge d_G(o, z_2)$, if $S_2(z_1)$ intersects $S_0(w)$, then $S_2(z_1)$ must be contained in the union of $S_2(z_2)$ and $S_2(z_3)$. The same holds in the second case, unless $d_G(o, z_1) = d_G(o, z_2)$. In this last case, one of the following three holds: (i) $S_2(z_1) \subset S_2(z_3)$, (ii) $S_2(z_1) = S_2(z_2)$, (iii) $S_2(z_2) \subset S_2(z_3) \cup S_2(z_1)$. In all three cases, the claim holds, hence the lemma. □

An extension of the results in this paper to higher complex dimensions is in N. ARCOZZI, R.ROCHBERG, E. SAWYER, "Carleson Measures and Interpolating Sequences for Besov Spaces on Complex Balls", to appear in *Memoirs of the A.M.S.* □

The covering lemma might also be proved taking into account the interpretation of the graph elements as Whitney boxes, then using elementary geometry.

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