Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 2, Article 28, 2003

GENERALIZED QUASI-VARIATIONAL INEQUALITIES AND DUALITY

JACQUELINE MORGAN AND MARIA ROMANIELLO

DIPARTIMENTO DI MATEMATICA E STATISTICA
UNIVERSITÀ DI NAPOLI FEDERICO II
VIA CINTHIA, 80126 NAPOLI, ITALY.
morgan@unina.it

DIPARTIMENTO DI ORGANIZZAZIONE AZIENDALE E AMMINISTRAZIONE PUBBLICA
UNIVERSITÀ DELLA CALABRIA
VIA PIETRO BUCCI, 87036 COSENZA, ITALY.
mromanie@unina.it

Received 18 November, 2002; accepted 4 February, 2003
Communicated by A.M. Rubinov

ABSTRACT. We present a scheme which associates to a generalized quasi-variational inequality a dual problem and generalized Kuhn-Tucker conditions. This scheme allows to solve the primal and the dual problems in the spirit of the classical Lagrangian duality for constrained optimization problems and extends, in non necessarily finite dimentional spaces, the duality approach obtained by A. Auslender for generalized variational inequalities. An application to social Nash equilibria is presented together with some illustrative examples.

Key words and phrases: Generalized quasi-variational inequality, Primal and dual problems, Generalized Kuhn-Tucker conditions, Banach space, Social Nash equilibrium, Subdifferential.

2000 *Mathematics Subject Classification.* 65K10, 49N15, 91A10.

1. INTRODUCTION

Let X be a real Banach space with dual X^* or, more generally, let X and X^* be two real locally convex topological vector spaces, duals with respect to a product of duality $\langle \cdot, \cdot \rangle$ (see [14, p. 336]).

If A and K are two set-valued operators from X to X^* and from X to X, respectively, we are interested to the following variational problem (in short (VP)):

$$(VP) \qquad \text{find } x^* \in X \text{ such that } x^* \in K(x^*) \text{ and there exists}$$
$$z^* \in A(x^*) \text{ satisfying } \langle z^*, x - x^* \rangle \geq 0, \quad \text{for all } x \in K(x^*).$$

This problem, called Generalized Quasi-Variational Inequality ([16], [8], [12], ...), generalizes the following problems:

– variational inequalities as introduced by G. Stampacchia [17] (see also [2], [6], [11], ...)

ISSN (electronic): 1443-5756

126-02

– generalized variational inequalities ([2], [5], [11], ...)
– quasi-variational inequalities ([6], [12], ...)

and describes various economic and engineering problems (see Section 3 and, for example, [1], [7], [10]).

Existence results for solutions of such a problem have been given in [8] and [16], while stability of the following problem (equivalent to (VP) under suitable assumptions):

$$(VP)': \text{ find } x^* \in X \text{ such that } x^* \in K(x^*) \text{ and}$$
$$\inf_{z^* \in A(x^*)} \langle z^*, x - x^* \rangle \geq 0, \quad \text{for all } x \in K(x^*)$$

has been investigated in [12].

Differently, to our knowledge there exists no results concerning a duality scheme or a numerical method which solves a generalized quasi-variational inequality. Nevertheless, in the case of generalized variational inequalities, for constraints defined by a finite number of inequalities and in finite dimensional spaces, A. Auslender introduced in [2] a duality scheme which associates to the Primal Problem another generalized variational inequality (with only constraints of positivity) for which an algorithm has been developed (see [3]).

In this paper, we extend to generalized quasi-variational inequalities in non necessarily finite dimensional spaces the duality approach obtained by Auslender for generalized variational inequalities. More precisely we present a scheme which associates to the variational problem (VP):

– a dual problem, called (DVP)
– Generalized Kuhn-Tucker Conditions

which allows us to solve (VP) and (DVP) in the spirit of the classical Lagrangian duality for constrained optimization problems. From a numerical point of view, we point out that the dual problem (DVP) has a special structure which allows to apply the algorithm introduced in [3] for generalized variational inequalities.

In Section 2, we present the duality scheme and the connections between the primal and the dual problems through the Generalized Kuhn-Tucker Conditions. In Section 3, we apply this method to find Social Nash Equilibria for nonzero-sum games with coupled constraints defined by a finite number of inequalities and we give some illustrative examples.

2. DUALITY SCHEME FOR (VP)

The scheme presented in this section takes advantage of the particular structure of the set-valued operator K defined by a finite number of inequalities. More precisely, we assume that for all $x \in X$:

$$K(x) = \{z \in X / f_j(x, z) \leq 0, \text{ for all } j = 1, 2, \ldots, m\}$$

where:

$(H1)$ $$f_j(x, \cdot) : X \to \mathbb{R} \cup \{+\infty\} \text{ is a proper, closed and convex function ([18]) for all } j = 1, \ldots, m.$$

Now, for all $u \in \mathbb{R}^m_+$, let

$$F(x, y) = (f_1(x, y), \ldots, f_m(x, y)) \tag{2.1}$$

and

$$G(u) = \left\{ -F(x, x) \; / \; 0 \in A(x) + \sum_{j=1}^{m} u_j \partial_2 f_j(x, x) \right\} \tag{2.2}$$

where $\partial_2 f_j(x,t)$ is the subdifferential of the function $f_j(x,\cdot)$ at the point t, that is:

$$\partial_2 f_j(x,t) = \{z \in X^*/ f_j(x,y) \geq f_j(x,t) + \langle z, y-t\rangle \, \forall y \in X\}$$

Definition 2.1. The *Dual Problem* of the problem (VP) (in short (DVP)), is the following generalized variational inequality:

(DVP) to find $u^* \in \mathbb{R}^m_+$ such that there exists $d^* \in G(u^*)$ satisfying $\langle d^*, u-u^*\rangle \geq 0, \quad$ for all $u \in \mathbb{R}^m_+$.

The problem (DVP) is termed a Dual Problem because we have:

Theorem 2.1. *Assume that ($H1$) is satisfied and that x^* is a point of X such that $E(x^*) = \cap_{j=1}^m \operatorname{dom}(f_j(x^*,\cdot))$ is an open subset of X. If (x^*, u^*), with $u^* \in \mathbb{R}^m_+$, satisfies the following conditions, called "Generalized Kuhn-Tucker Conditions":*

$(KT)_1$: $x^* \in K(x^*)$;
$(KT)_2$: $0 \in A(x^*) + \sum_{j=1}^m u_j^* \partial_2 f_j(x^*, x^*)$;
$(KT)_3$: $F(x^*, x^*) \in N_{\mathbb{R}^m_+}(u^*)$;

then

(i) x^* *is a solution to (VP)*
(ii) u^* *is a solution to (DVP).*

Proof. First, to prove (i) we observe that:

"(x^*, z^*), with $z^* \in A(x^*)$, solves (VP)"

is equivalent to

"x^* is a solution to the optimization problem (OP)"

where (OP) is:

(OP)
$$\min_{x \in K(x^*)} \langle z^*, x - x^*\rangle .$$

The problem (OP) admits as classical Lagrangian the function L, from $E(x^*) \times \mathbb{R}^m$ to $\overline{R}$, defined by:

$$L(x,u) = \begin{cases} \langle z^*, x-x^*\rangle + \sum_{j=1}^m u_j f_j(x^*, x) & \text{if } x \in E(x^*) \text{ and } u \in \mathbb{R}^m_+ \\ -\infty & u \notin \mathbb{R}^m_+ \\ +\infty & \text{otherwise.} \end{cases}$$

So to prove (i), it is sufficient to apply the Theorem 7.5.1 in ([14]) to the problem (OP), taking into account that $N_{E(x^*)}(x^*) = \{0\}$ (since $E(x^*)$ is open) and $\partial(\langle z^*, x-x^*\rangle) = z^* + N_{E(x^*)}(x^*)$.

Now we prove (ii). In light of the assumption $(KT)_2$, it follows that $-F(x^*, x^*) \in G(u^*)$, where F and G are defined, respectively, by (2.1) and (2.2). So, since $F(x^*, x^*) \in N_{\mathbb{R}^m_+}(u^*)$ by assumption $(KT)_3$, and

$$N_{\mathbb{R}^m_+}(u^*) = \begin{cases} \{v \in \mathbb{R}^m_+ / \langle v, u-u^*\rangle \leq 0 \quad \forall u \in \mathbb{R}^m_+\} & \text{if } u^* \in \mathbb{R}^m_+ \\ \emptyset & \text{otherwise,} \end{cases}$$

then u^* solves the problem (DVP) defined in Definition 2.1. □

Theorem 2.2. *Assume that ($H1$) is satisfied. If x^* is a solution to (VP) and if:*

(i) $E(x^*) = \cap_{j=1}^m \operatorname{dom}(f_j(x^*,\cdot))$ *is an open subset of* X
(ii) $\exists y \in X$ *such that* $f_j(x^*, y) < 0$ *for all* $j = 1, \ldots, m$

then, there exists a point $u^* \in \mathbb{R}_+^m$ such that (x^*, u^*) satisfies the Generalized Kuhn-Tucker Conditions $(KT)_1$ to $(KT)_3$ (and therefore u^* solves (DVP) following Theorem 2.1).

Proof. Let x^* be a solution to (VP) and $z^* \in A(x^*)$ such that $\langle z^*, x - x^* \rangle \geq 0$ for all $x \in K(x^*)$. By Theorem 7.5.2 in [14], there exists a point $u^* \in \mathbb{R}_+^m$ such that (x^*, u^*) is a saddle point for the Lagrangian L above defined. So, it results that:

$$0 \in \partial_x L(x^*, u^*) = z^* + \sum_{j=1}^{m} u_j^* \partial_2 f_j(x^*, x^*)$$

which implies that $0 \in A(x^*) + \sum_{j=1}^{m} u_j^* \partial_2 f_j(x^*, x^*)$. Moreover, since $L(x^*, u) \leq L(x^*, u^*)$ for all $u \in \mathbb{R}_+^m$:

$$\sum_{j=1}^{m} \left(u_j - u_j^*\right) f_j(x^*, x^*) = \langle F(x^*, x^*), u - u^* \rangle \leq 0 \;\; \forall u \in \mathbb{R}_+^m$$

that is $F(x^*, x^*) \in N_{\mathbb{R}_+^m}(u^*)$. Therefore (x^*, u^*) satisfies $(KT)_1$ to $(KT)_3$ and u^* solves (DVP). □

In light of Theorems 2.1 and 2.2, the variational problem (DVP) can be considered as a dual problem associated to (VP).

Remark 2.3. If $X = \mathbb{R}^n$, for all $x \in X$:

$$K(x) = C = \{z \in X / f_j(z) \leq 0, \text{ for all } j = 1, 2, \ldots, m\}$$

and the generalized quasi-variational inequality comes from an optimization problem defined by a convex and differentiable function, then the previous theorems reduce to the classical theorems of Convex Mathematical Programming (Theorems 3.2 and 3.3 in [2]).

Remark 2.4. Let us observe that the condition

$$E(x^*) = \cap_{j=1}^{m} \operatorname{dom}(f_j(x^*, \cdot)) \text{ is an open set of } X$$

has been needed to properly handle convex programs within the formalism of extended valued functions ([14]).

By the previous theorems it follows that, to solve (VP), one can solve the dual problem (DVP) and then, using the generalized Kuhn-Tucker condition $(KT)_2$, one can find the solutions of problem (VP) proceeding as in the following example.

Example 2.1. If

$$K(x) = \{y \in \mathbb{R} / y - 2x \leq 0 \text{ and } x - y \leq 0\}$$

and

$$A(x) = \begin{cases} \left[x - \frac{1}{3}, 0\right[& \text{if } 0 < x < \frac{1}{3} \\ [x, 1] & \text{if } \frac{1}{3} \leq x \leq 1 \\ \emptyset & \text{otherwise} \end{cases}$$

then the dual problem (DVP) associated to the primal problem (VP) is the easier generalized variational inequality:

$$\text{to find } u^* \in \mathbb{R}_+^2 \text{ such that there exists } d^* \in G(u^*)$$
$$\text{satisfying } \langle d^*, u - u^* \rangle \geq 0, \quad \text{for all } u \in \mathbb{R}_+^2$$

where

$$G(u_1,u_2)=\begin{cases} \left]0,u_2-u_1+\frac{1}{3}\right]\times\{0\} & \text{if } -\frac{1}{3}<u_2-u_1<0\\ \left[\frac{1}{3},u_2-u_1\right]\times\{0\} & \text{if } \frac{1}{3}\le u_2-u_1\le 1\\ \emptyset & \text{otherwise.}\end{cases}$$

The solutions to the problem (DVP) are all the points $(0,u_2)$ such that 1/3 $\le u_2\le 1$, so, using the Generalized Kuhn-Tucker Condition $(KT)_2$, we find that all the points x^* such that 1/3 $\le x^*\le 1$ are solutions to (VP).

3. APPLICATION TO SOCIAL NASH EQUILIBRIA

Let us consider a n-person noncooperative game with coupled constraints, as considered by G. Debreu in [7]. Let Y_i be a Banach space (or, more generally, a real locally convex topological vector space) and, for the player i, let $X_i\subseteq Y_i$ be the strategy set, J_i from $X=X_1\times\cdots\times X_n$ to $\mathbb{R}$ be the payoff function, and

$$K_i(x_{-i})=\left\{y_i\in X_i/f_j^i(y_i,x_{-i})\le 0,\text{for all } j=1,2,\ldots,m_i\right\}$$

be the constraints depending on the strategies of the other players, where x_{-i} is a shorthand for $(x_j)_{j\in N\setminus\{i\}}$. We assume that the players want to minimize their payoff function and play a Social Nash Equilibrium [7] (also called Generalized Nash Equilibrium [10], which is a generalization of the concept of Nash Equilibria [15]). We recall that a Social Nash Equilibrium of the game $\Gamma=\{X_i,J_i,K_i\}$ is a point $x^*\in X$ such that no player can uniterally decrease his payoff given the constraints imposed on him by the other players; that is a point such that:

(SNE) $\qquad J_i(x^*)\le J_i\left(x_i,x_{-i}^*\right)$ for all $x_i\in K_i\left(x_{-i}^*\right)$ and for all $i=1,\ldots n$.

It is well known that, under suitable assumptions, the Social Nash Equilibrium problem can be put into the form of a generalized quasi-variational inequality (see for example [6, 4, 11]). More precisely, if we assume that the following condition is satisfied:

(*H2*) for every $x_{-i}\in X_{-i}$ the function $J_i(\cdot,x_{-i})$ is convex and bounded from below on X_i, for all $i=1,\ldots,n$

then, a point x^* is a solution to the problem (SNE) if and only if x^* solves the following system of generalized quasi-variational inequalities:

$$(SNE)\qquad\begin{cases} \text{find } x^*\in X \text{ such that } x^*\in K_1\left(x_{-1}^*\right)\times\cdots\times K_n\left(x_{-n}^*\right)\\ \text{and there exist } z_1^*\in\partial_{x_1}J_1(x^*),\cdots,z_n^*\in\partial_{x_n}J_n(x^*) \text{ satisfying}\\ \langle z_1^*,x_1-x_1^*\rangle\ge 0,\quad \text{for all } x_1\in K_1\left(x_{-1}^*\right)\\ \vdots\\ \langle z_n^*,x_n-x_n^*\rangle\ge 0,\quad \text{for all } x_n\in K_n\left(x_{-n}^*\right)\end{cases}$$

where $\partial_{x_i}J_i$ is the subdifferential of $J_i(\cdot,x_{-i})$ for all $i=1,\ldots,n$.

Now, if we considered the set-valued operator defined on X by:

$$A(x)=\partial_{x_1}J_1(x)\times\cdots\times\partial_{x_n}J_n(x)$$

and

$$\begin{aligned} K(x) &= \{y\in X\ /\ y_i\in K_i(x_{-i})\,\forall i=1,\ldots,n\}\\ &= \{y\in X\ /\ f_j(x,y)\le 0\ \ j=1,\ldots,m\}\end{aligned}$$

where $m = m_1 + \cdots + m_n$ and

$$f_j(x,y) = \begin{cases} f_j^1(y_1, x_{-1}) & \text{if } j = 1, \ldots, m_1 \\ \vdots & \\ f_j^i(y_i, x_{-i}) & \text{if } j = \sum_{r=1}^{i-1} m_r + 1, \ldots, \sum_{r=1}^{i-1} m_r + m_i \\ \vdots & \\ f_j^n(y_n, x_{-n}) & \text{if } j = \sum_{r=1}^{n-1} m_r + 1, \ldots, m \end{cases}$$

then x^* is a Social Nash Equilibrium for Γ if and only if it solves the following generalized quasi-variational inequality:

(SNE) find $x^* \in X$ such that $x^* \in K(x^*)$ and there exists

$z^* \in A(x^*)$ satisfying $\langle z^*, x - x^* \rangle \geq 0,$ for all $x \in K(x^*)$.

If the problem (SNE) satisfies the assumptions *(H1)* and *(H2)*, we can define the dual problem:

$(DSNE)$ find $u^* \in \mathbb{R}_+^m$ such that there exists $d^* \in G(u^*)$

satisfying $\langle d^*, u - u^* \rangle \geq 0,$ for all $u \in \mathbb{R}_+^m$,

where G is the set-valued operator defined by:

$$G(u) = \left\{ -F(x,x) \,/\, 0 \in \partial_{x_h} J_h(x) + \sum_{j=1}^{m} u_j \partial_{x_h} f_j(x,x), \text{ for all } h = 1, \ldots, n \right\}.$$

Therefore, we can find the Social Nash equilibria of Γ using the method introduced in section 2, as one can see in the following example:

Example 3.1. Let us consider a two-player game Γ with

$$J_1(x,y) = x^2 + 2x - y^2$$

$$J_2(x,y) = y^2 + 2xy$$

and

$$K_1(y) = \{x \in \mathbb{R} \,/\, x - y \leq 0\}$$

$$K_2(x) = \{y \in \mathbb{R} \,/\, 2x - y \leq 0\} \ .$$

The Social Nash Equilibrium problem associated to this game is equivalent to the following generalized quasi-variational inequality:

(SNE) find $(x^*, y^*) \in K(x^*, y^*)$

such that $(2x^* + 2)(x - x^*) + (2y^* + 2x^*)(y - y^*) \geq 0$

for all $(x,y) \in K(x^*, y^*)$.

Since

$$G(u_1, u_2) = \{(2u_1 + u_2 + 4/2, 3u_1 + u_2 + 6/2)\},$$

the dual of (SNE) is the easier problem:

$(DSNE)$ find $u^* \in \mathbb{R}_+^2$ such that

$$(2u_1^* + u_2^* + 4/2)(u_1 - u_1^*) + (3u_1^* + u_2^* + 6/2)(u_2 - u_2^*) \geq 0$$

for all $u \in \mathbb{R}_+^2$.

The unique solution of ($DSNE$) is $(u_1^*, u_2^*) = (0,0)$ and so, by the Generalized Kuhn-Tucker Condition $(KT)_2$, we have that the point $(x^*, y^*) = (-1,1)$ is a Social Nash Equlibrium for the game Γ.

REFERENCES

[1] C.D. ALIPRANTIS, D. BROWN AND O. BURKINSHAW, *Existence and Optimality of Competitive Equilibria*, Springer Verlag, Berlin-Heidelbrg-New York-London-Paris-Tokyo, (1988).

[2] A. AUSLENDER, *Optimisation. Méthodes Numériques*, Masson, Paris-New York-Barcelona, (1976).

[3] A. AUSLENDER AND M. TEBOULLE, Lagrangian duality and related multiplier methods for variational inequality problems, *SIAM J. Optim.*, **10**(4) (2000), 1097–1116.

[4] A. BENSOUSSAN, Points de Nash dans le cas de fonctionnelles quadratiques et jeux differentials linéaires a N personnes, *SIAM J. on Control*, **12** (1974), 460-499.

[5] J.P. CROUZEIX, Pseudomonotone variational inequality problems: Existence of solutions, *Math. Program.*, **78** (1997), 305–314.

[6] C. BAIOCCHI AND A. CAPELO, *Disequazioni Variazionali e Quasivariazionali. Applicazioni a Problemi di Frontiera Libera*, Quadernidell' U.M.I., Pitagora Editrice, Bologna, (1978).

[7] G. DEBREU, A social equilibrium existence theorem, *Proc. Nat. Acad. Sci.*, USA, **38** (1952), 886–893.

[8] X.P. DING AND K.K. TAN, Generalized variational inequalities and generalized quasi-variational inequalities, *J. Math. Anal. Appl.*, **148** (1990), 497–508.

[9] P.T. HARKER AND J.S. PANG, Finite-dimensional variational inequality and non linear complementarity problems: a survey of theory, algorithms and applications, *Math. Program.*, **48** (1990), 171–220.

[10] T. ICHIISHI, *Game Theory for Economic Analysis*, Academic Press, New York, (1983).

[11] M.B. LIGNOLA AND J. MORGAN, Generalized variational inequalities with pseudomonotone operators under perturbations, *J. Optim. Theory Appl.*, **101** (1999), 213–220.

[12] M.B. LIGNOLA AND J. MORGAN, Convergence of solutions of quasi-variational inequalities and applications, *Topol. Methods Nonlinear Anal.*, **10** (1997), 375–385.

[13] M.B. LIGNOLA AND J. MORGAN, Approximate solutions and alpha-well-posedness for variational inequalities and Nash equilibria, in *Decision and Control in Management Science*, Kluwer Academic Science, (2002), 367–378.

[14] P.J. LAURENT, *Approximation and Optimisation*, Hermann, Paris-London,(1972).

[15] J.F. NASH Jr., Equilibrium points in n-person games, *Proc. Nat. Acad. Sci., USA*, **36** (1950), 48–49.

[16] M.H. SHIH AND K.K. TAN, Generalized quasi-variational inequalities in locally convex vector spaces, *J. Math. Anal. Appl.*, **108** (1985), 333–343.

[17] G. STAMPACCHIA, Variational inequalities, in theory and applications of monotone pperators, *Proc. NATO Advanced Study Inst.*, Edizioni Oderisi, (1968), 101–192.

[18] J.V. TIEL, *Convex Analysis. An Introduction Text*, John Wiley and Sons, (1984).