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MARTINGALE INEQUALITIES IN EXPONENTIAL ORLICZ SPACES

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ABSTRACT. A result is found which is similar to BDG-inequalities, but in the framework of exponential (non moderate) Orlicz spaces. A special class of such spaces is introduced and its properties are discussed with respect to probability measures, whose densities are connected by an exponential model.

Key words and phrases: Orlicz space, BDG-inequalities, exponential model.

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1. INTRODUCTION

Exponential martingale inequalities are a very important and still relevant topic in Martingale Theory: see e.g. [5], [10], [11] and [9] for recent literature. In particular, inequalities involving a continuous martingale and its quadratic variation are considered in [10] and [5].

An attempt has been made to find exponential inequalities that relate a generic continuous martingale and its quadratic variation by investigating results similar to Burkholder, Davis and Gundy's (BDG) inequalities, but in the framework of exponential (non moderate) Orlicz spaces. A first attempt on this topic can be found in [6], where exponential BDG-type inequalities are discussed for a Brownian motion.

The analytical framework of (exponential) Orlicz spaces has recently been given renewed relevance - see e.g. [1] and [12] - and may have applications in the field of Mathematical Finance. For instance, semimartingales such that their quadratic variation belongs to the exponential Orlicz space are considered in [17]. Moreover, a general Orlicz space based approach for utility maximization problems is described in [2] and [3]. However, BDG inequalities are interesting in themselves. For instance, BDG-type inequalities are used in [18] to find closure properties in Lebesgue spaces that are directly related to variance-optimal hedging strategies.

In order to state our results, a special class of exponential Orlicz spaces is introduced and its properties are discussed in relation to different probability measures.

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More precisely, in Section 2 we analyze in detail the structure of exponential Orlicz spaces by defining the class of L^{n,Φ_1} spaces as the sets of random variables whose n-power belongs to L^{Φ_1}, where $\Phi_1(x) = \cosh(x) - 1$. Such discussions are generalizations of previous results based on [15], [14] and [4], regarding the topology of L^{Φ_1} and its applications to exponential models. In particular, we study the equivalence of norms among these spaces with respect to different probability measures, whose densities are connected by an open exponential arc.

The main result is given in Section 3, where BDG-type inequalities are discussed within the topology of L^{n,Φ_1} spaces, with respect to different measures. Finally, we show that such measures are connected by an open exponential arc and therefore the corresponding spaces have equivalent norms.

2. EXPONENTIAL ORLICZ SPACES

2.1. **Analytical framework.** Before showing the main results of this paper, a brief introduction to Orlicz spaces is necessary: reference can be made to [16] for the general theory and to [15], [14] and [4] for connections to exponential models.

Let us fix a probability space $(\Omega, \mathcal{F}, \mu)$ and let $\mathcal{D}(\Omega, \mathcal{F}, \mu)$ be the set of the μ-almost surely strictly positive densities. Let $L^\Phi(\mu)$ be the Orlicz space associated to the Young function Φ: it can be proved that $L^\Phi(\mu)$ is a Banach space endowed with the Luxemburg norm

$$||u||_{(\Phi,\mu)} = \inf\{k > 0 : \mathbb{E}[\Phi(u/k)] \leq 1\}. \tag{2.1}$$

It is possible to characterize functions that belong to the closed unit ball of $L^\Phi(\mu)$ using the following property - see e.g. [16, p. 54]

$$||u||_{(\Phi,\mu)} \leq 1 \Longleftrightarrow \mathbb{E}[\Phi(u)] \leq 1. \tag{2.2}$$

Furthermore, this norm is monotone, that is, $|u| \leq |v|$ implies $||u||_{(\Phi,\mu)} \leq ||v||_{(\Phi,\mu)}$.

From now on, we shall deal with the space $L^{\Phi_1}(\mu)$ associated with the function $\Phi_1(x) := \cosh(x) - 1$. Let $\Psi_1(x) := (1 + |x|)\log(1 + |x|) - |x|$ be the conjugate function of $\hat{\Phi}(x) := \exp(|x|) - |x| - 1$. Since Φ_1 and $\hat{\Phi}$ are equivalent Young functions, we shall refer to Ψ_1 as the conjugate of Φ_1 in the sequel.

The following result will be used hereafter.

Proposition 2.1 (see [14]). *Let $p, q \in \mathcal{D}(\Omega, \mathcal{F}, \mu)$ be connected by a one-dimensional open exponential model. More precisely, let $r \in \mathcal{D}(\Omega, \mathcal{F}, \mu)$ and $u \in L^{\Phi_1}(r \cdot \mu)$ and let us suppose that there exists an exponential model*

$$p(\theta, x) := e^{\theta u(x) - \psi(\theta)} r(x), \tag{2.3}$$

where $\theta \in (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$, for some positive ε and $\psi(\theta)$ is the cumulant generating function, such that $p(\theta_0) = p$ and $p(\theta_1) = q$. Then $L^{\Phi_1}(p \cdot \mu)$ and $L^{\Phi_1}(q \cdot \mu)$ are equal as sets and have equivalent norms.

2.2. **The space L^{n,Φ_1}.** The topology of $L^{\Phi_1}(\mu)$ is a natural framework to consider the *moment generating functional* $\mathbb{E}[e^u]$ of a random variable u. More generally, let us also take into account the moment generating functional of powers u^n, where $n \geq 1$. For this purpose, we introduce a more general class of Orlicz spaces.

For $n \geq 1$, let us define

$$L^{n,\Phi_1}(\mu) := \{u : u^n \in L^{\Phi_1}(\mu)\}; \tag{2.4}$$

it is trivial to show that $L^{n,\Phi_1}(\mu)$ is a subspace of $L^{\Phi_1}(\mu)$, because $|u| \leq 1 + |u|^n$ for each real number u.

In fact, $L^{n,\Phi_1}(\mu)$ is an Orlicz space with respect to the Young function $\Phi_n(x) := \cosh(x^n)-1$. Therefore, we can endow it with the usual norm: given $u \in L^{n,\Phi_1}(\mu)$, we have

$$||u||_{(\Phi_n,\mu)} := \inf\{r > 0 : \ \mathbb{E}[\exp(u^n)] + \mathbb{E}[\exp(-u^n)] \leq 4\}. \tag{2.5}$$

An easy computation shows that these norms are related to the topology of $L^{\Phi_1}(\mu)$ through the following equality

$$||u||_{(\Phi_n,\mu)} = ||u^n||^{\frac{1}{n}}_{(\Phi_1,\mu)}. \tag{2.6}$$

Unfortunately, the conjugate function of $\Phi_n(x)$ does not simply admit an explicit expression. However, if we define $\phi_n(x) := nx^{n-1}\sinh(x^n)$, a straight integration gives the following expression for the conjugate $\Psi_n(x)$

$$\Psi_n(x) = n(\phi_n^{-1}(x))^n \sinh((\phi_n^{-1}(x))^n) - \cosh((\phi_n^{-1}(x))^n) + 1. \tag{2.7}$$

Since $\cosh(x^n) \leq \cosh(x^m)$ for any $m \geq n \geq 1$ and $x \geq 1$, from e.g. [16, p. 155] one obtains

$$L^{m,\Phi_1}(\mu) \subset L^{n,\Phi_1}(\mu), \tag{2.8}$$

for any $m \geq n \geq 1$. More precisely, these inclusions correspond to continuous embedding of one space into another, that is, for any $m \geq n \geq 1$ there exists a positive constant $k := 1 + \Phi_n(1)\mu(\Omega) = (e^2+1)/2e$ such that

$$||u||_{(\Phi_n,\mu)} \leq k||u||_{(\Phi_m,\mu)}. \tag{2.9}$$

It is natural to consider the intersection of such spaces: for this purpose, let us define

$$L^{\infty,\Phi_1}(\mu) := \bigcap_{n\geq 1} L^{n,\Phi_1}(\mu). \tag{2.10}$$

First of all, note that L^{∞,Φ_1} is not empty, since it contains all the bounded functions. Moreover, since the product uv can be upper bounded by the sum u^2+v^2, it can be shown that $L^{\infty,\Phi_1}(\mu)$ is an algebra.

At this point, it is possible to ask whether, in general, $L^{\infty,\Phi_1}(\mu)$ and $L^{\infty}(\mu)$ are equal as sets.

Proposition 2.2. *Let μ be the Lebesgue measure on $[0,1]$; then $L^{\infty}(\mu)$ is strictly included in $L^{\infty,\Phi_1}(\mu)$.*

Proof. Let us define

$$u(x) := \log(1-\log(x)) \tag{2.11}$$

and fix $n \geq 1$ and $r < 1$. Trivially, $\mathbb{E}[\exp(-ru^n)] < \infty$; let us study the convergence of $\mathbb{E}[\exp(ru^n)]$. For any x belonging to a suitable neighborhood of zero, the following holds

$$u(x) \leq [1-\log(x)]^{\frac{1}{n}}, \tag{2.12}$$

and hence

$$\exp(ru^n) \leq e^r \exp(-r\log(x)). \tag{2.13}$$

Since $\mathbb{E}[\exp(-r\log(x))] < \infty$, we can conclude that $u \in L^{n,\Phi_1}(\mu)$, proving the thesis. ☐

We conclude this section by investigating relationships among L^{n,Φ_1} spaces with respect to different probability measures. Such a result will be useful to better understand the structure of the Burkholder-type inequalities that will be discussed in the next section. The proof is a consequence of [4, Lemma 18, p. 40].

Proposition 2.3. *For each $p, q \in \mathcal{D}$ connected by a one-dimensional open exponential model, $L^{n,\Phi_1}(p\cdot\mu)$ and $L^{n,\Phi_1}(q\cdot\mu)$ are equal as sets and have equivalent norms.*

Remark 1. It should be noted that the definition of L^{n,Φ_1} and its basic properties are similar to the theory of classical Lebesgue spaces L^p.

From now on, we shall limit our study to the space L^{2,Φ_1}. The following theorem states the continuity of the product uv in L^{Φ_1}.

Theorem 2.4. *Let $p \geq 1$ and q be its conjugate; let us consider $u \in L^{p,\Phi_1}(\mu)$ and $v \in L^{q,\Phi_1}(\mu)$; then*

$$||uv||_{(\Phi_1,\mu)} \leq ||u||_{(\Phi_p,\mu)}||v||_{(\Phi_q,\mu)}. \tag{2.14}$$

Proof. Let $s := ||u^p||_{(\Phi_1,\mu)}$, $m := ||v^q||_{(\Phi_1,\mu)}$, $\varepsilon := (s/m)^{\frac{1}{pq}}$ and $r := s^{\frac{1}{p}} m^{\frac{1}{q}}$; from the inequality

$$uv \leq \frac{1}{p}\frac{u^p}{\varepsilon^p} + \frac{1}{q} v^q \varepsilon^q \tag{2.15}$$

and by using the convexity of Φ_1 we obtain

$$\begin{aligned} \mathbb{E}\left[\Phi_1\left(\frac{uv}{r}\right)\right] &\leq \frac{1}{p}\mathbb{E}\left[\Phi_1\left(\frac{u^p}{r\varepsilon^p}\right)\right] + \frac{1}{q}\mathbb{E}\left[\Phi_1\left(\frac{v^q\varepsilon^q}{r}\right)\right] \\ &\leq \frac{1}{p}\mathbb{E}\left[\Phi_1\left(\frac{u^p}{s}\right)\right] + \frac{1}{q}\mathbb{E}\left[\Phi_1\left(\frac{v^q}{m}\right)\right] \\ &\leq \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \tag{2.16}$$

Therefore, the following holds

$$||uv||_{(\Phi_1,\mu)} \leq r = ||u^p||^{\frac{1}{p}}_{(\Phi_1,\mu)} ||v^q||^{\frac{1}{q}}_{(\Phi_1,\mu)}, \tag{2.17}$$

and (2.6) gives the inequality we were looking for. □

More generally, a standard argument shows the following corollary.

Corollary 2.5. *The function $F : L^{2,\Phi_1}(\mu) \ni u \mapsto u^2 \in L^{\Phi_1}(\mu)$ is continuous; furthermore, it is Fréchet differentiable and its differential dF evaluated at the point u in the direction v is equal to $dF(u)[v] = 2uv$.*

Moreover, from Theorem 2.4 and since the topology of L^{Φ_1} is stronger that any L^p space, the following statement can be easily proved.

Corollary 2.6. *The scalar product $\langle u, v\rangle_{L^2} := \mathbb{E}[uv]$ is continuous in $L^{2,\Phi_1}(\mu) \times L^{2,\Phi_1}(\mu)$.*

3. MARTINGALE INEQUALITIES WITHIN L^{n,Φ_1} SPACES

Let $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_t)$, where $t \in [0, T]$ and $T < \infty$, be a filtered probability space that satisfies the usual conditions. From now on, we shall consider adapted processes with continuous trajectories and denote the space of continuous martingales starting from zero with $\mathcal{M}_c$.

For the sequel, it is useful to reformulate a classical sufficient condition in the topology of $L^{n,\Phi}(\mu)$ spaces which can ensure that the so-called *exponential martingale*

$$Z_t := \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right) := \mathcal{E}_t(M), \tag{3.1}$$

where M is a local martingale, is a true martingale. If this is the case, $\mathcal{E}_t(M)$ is actually a *Girsanov density* for any $t \in [0, T]$. However, in the general case Z is a supermartingale, so that $\mathbb{E}[Z_t] \leq 1$ for each t. For a deeper insight into these topics, reference can be made to [8]. In particular, in [8, p. 8] it is proved that Z is a martingale if there exists a $m > 1$ such that

$$\sup_{\tau \leq T} \mathbb{E}\left[\exp\left(\frac{\sqrt{m}}{2(\sqrt{m}-1)} M_\tau\right)\right] < \infty. \tag{3.2}$$

Proposition 3.1. *Let $M \in \mathcal{M}_c$ be a continuous martingale such that $||M_T||_{(\Phi_1,\mu)} < 2$. Then $\mathcal{E}(M)$ is a martingale.*

Proof. Since $||M_T||_{(\Phi_1,\mu)} < 2$, there exists a $\beta > 0$ such that

$$\frac{1}{\beta} = \frac{\sqrt{m}}{2(\sqrt{m}-1)}, \tag{3.3}$$

for some $m \in (1, \infty)$. Moreover, $||M_T/\beta||_{(\Phi_1,\mu)} \leq 1$, so that, from (2.2),

$$\mathbb{E}\left[\exp\left(\frac{1}{\beta} M_T\right)\right] \leq 4 < \infty. \tag{3.4}$$

Since $M \in \mathcal{M}_c$, due to the convexity of $\Phi_1(x)$, for any stopping time $\tau \leq T$

$$||M_\tau||_{(\Phi_1,\mu)} \leq ||M_T||_{(\Phi_1,\mu)}. \tag{3.5}$$

Therefore

$$\sup_{\tau \leq T} \mathbb{E}\left[\exp\left(\frac{\sqrt{m}}{2(\sqrt{m}-1)} M_\tau\right)\right] = \sup_{\tau \leq T} \mathbb{E}\left[\exp\left(\frac{1}{\beta} M_\tau\right)\right] \leq 4 < \infty. \tag{3.6}$$

□

3.1. **BDG-inequalities within L^{n,Φ_1} spaces.** Let $\Phi(t)$ be a Young function expressed in integral form as

$$\Phi(t) = \int_0^t \phi(s) ds; \tag{3.7}$$

define

$$\gamma := \sup_t \frac{t\phi(t)}{\Phi(t)} \tag{3.8}$$

and

$$\gamma' := \inf_t \frac{t\phi(t)}{\Phi(t)}. \tag{3.9}$$

The function Φ is said to be *moderate* if $\gamma < \infty$. For instance, $\Psi_1(x) = (1+|x|)\log(1+|x|) - |x|$, that is the conjugate function of $\Phi_1(x) = \cosh(x) - 1$, is moderate, since it has logarithmic form. Furthermore, when $\Phi = \Phi_1$ a straightforward computation shows that $\gamma' = 2$. Therefore, see e.g. [7, p. 186], the following generalized Doob's inequality can be stated in $L^{\Phi_1}(\mu)$.

Proposition 3.2. *Let $M \in \mathcal{M}_c$ and $M^* := \sup_{0 \leq s \leq T} |M_s|$; then*

$$||M^*||_{(\Phi_1,\mu)} \leq 2||M_T||_{(\Phi_1,\mu)}. \tag{3.10}$$

Given a local martingale M and a moderate Φ, Burkholder, Davis and Gundy's (BDG) classical inequalities are the following ones, see e.g. [7, p. 304]

$$\frac{1}{4\gamma}||M^*||_{(\Phi,\mu)} \leq \left\|\langle M\rangle_T^{\frac{1}{2}}\right\|_{(\Phi,\mu)} \leq 6\gamma||M^*||_{(\Phi,\mu)}. \tag{3.11}$$

When $\gamma = \infty$, (3.11) becomes meaningless, therefore different results could be expected.

In the sequel, we shall allow the norm of two different Orlicz spaces to appear in (3.11), provided they both belong to the exponential class L^{n,Φ_1}. In this way, we shall show that the former inequality in (3.11) still holds with a different constant, while the latter holds provided that different measures are allowed.

Proposition 3.3. *Let $M \in \mathcal{M}_c$ and $\tau \leq T$ be a stopping time; if $\langle M\rangle_T \in L^{\Phi_1}(\mu)$, then $M_\tau \in L^{\Phi_1}(\mu)$ and*

$$||M_\tau||_{(\Phi_1,\mu)} \leq \sqrt{2}\left\|\langle M\rangle_\tau^{\frac{1}{2}}\right\|_{(\Phi_2,\mu)}. \tag{3.12}$$

Therefore

$$||M^*||_{(\Phi_1,\mu)} \leq 2\sqrt{2}\left\|\langle M\rangle_T^{\frac{1}{2}}\right\|_{(\Phi_2,\mu)}. \tag{3.13}$$

Proof. Since $\langle M\rangle_T \in L^{\Phi_1}(\mu)$ and due to the monotonicity of the norm, $\langle M\rangle_\tau \in L^{\Phi_1}(\mu)$ for each $\tau \leq T$. Let $q := ||\langle M\rangle_\tau||_{(\Phi_1,\mu)} < \infty$ and define $r := \sqrt{2q}$. Using Hölder's inequality we obtain

$$\mathbb{E}\left[\exp\left(\pm\frac{1}{r}M_\tau\right)\right] \leq \left\{\mathbb{E}\left[\mathcal{E}_\tau\left(\pm\frac{2}{r}M\right)\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}\left[\exp\left(\frac{2}{r^2}\langle M\rangle_\tau\right)\right]\right\}^{\frac{1}{2}} \leq 2, \tag{3.14}$$

therefore

$$||M_\tau||_{(\Phi_1,\mu)} \leq r = \sqrt{2}||\langle M\rangle_\tau||^{\frac{1}{2}}_{(\Phi_1,\mu)} = \sqrt{2}\left\|\langle M\rangle_\tau^{\frac{1}{2}}\right\|_{(\Phi_2,\mu)}, \tag{3.15}$$

which provides (3.12). The inequality (3.13) is a consequence of Proposition 3.10. □

Remark 2. By definition of norm, from (3.13) one has

$$\mathbb{E}\left[\exp\left(\frac{M^*}{2\sqrt{2}\left\|\langle M\rangle_\tau^{\frac{1}{2}}\right\|_{(\Phi_2,\mu)}}\right)\right] \leq 4. \tag{3.16}$$

For instance, for a Brownian motion $(B_t)_{t\leq T}$, one obtains

$$\mathbb{E}\left[\exp\left(\frac{B_T^*}{2\sqrt{2T}}\right)\right] \leq 4. \tag{3.17}$$

Similar exponential inequalities are widely discussed in [6].

Theorem 3.4 (Main)**.** *Let $M \in \mathcal{M}_c$ be a non zero martingale such that $M_T \in L^{\Phi_1}(\mu)$, let $k \in (2-\sqrt{2}, 2]$ and $\tau \leq T$ be a stopping time such that $M_\tau \neq 0$. Then:*

- (i)**:** *$\langle M\rangle_\tau \in L^{\Phi_1}(q_{k\alpha_\tau}\cdot\mu)$, where $q_{k\alpha_\tau} := \mathcal{E}_\tau(k^{-1}M/\alpha_\tau)$ and $\alpha_\tau := ||M_\tau||_{(\Phi_1,\mu)}$. Furthermore, the following holds*

$$\left\|\langle M\rangle_\tau^{\frac{1}{2}}\right\|_{(\Phi_2,q_{k\alpha_\tau}\cdot\mu)} \leq \sqrt{c_k}||M_\tau||_{(\Phi_1,\mu)}, \tag{3.18}$$

where $c_k := 4k^2/(-2+4k-k^2)$;

- (ii)**:** *if $k=1$, we have the most stringent inequality and obtain*

$$\left\|\langle M\rangle_\tau^{\frac{1}{2}}\right\|_{(\Phi_2,q_{\alpha_\tau}\cdot\mu)} \leq 2||M_\tau||_{(\Phi_1,\mu)}. \tag{3.19}$$

Proof. Statement (ii) follows directly from (i) by minimizing the constant c_k with respect to k. Hence,it is only necessary to prove assertion (i).

Let us first show that (3.18) holds for $\tau \equiv T$. In order to prove this, we can suppose $\langle M\rangle_T \neq 0$; otherwise, the thesis is trivial.

Let us fix $k \in (2-\sqrt{2}, 2]$ and prove that $q_{k\alpha_T}$ is a density. By definition of α_T and since $k > \frac{1}{2}$ one obtains

$$||k^{-1}M_T/\alpha_T||_{(\Phi_1,\mu)} < 2. \tag{3.20}$$

Thus, from Proposition 3.1, $\mathcal{E}(k^{-1}M/\alpha_T)$ is a uniformly integrable martingale, so that $q_{k\alpha_T}$ is a density. Let $c_k := 4k^2/(-2+4k-k^2)$ and $r := c_k\alpha_T^2$ and define $1/s^2 := -1/r + 1/(2k^2\alpha_T^2)$; it should be noted that c_k is positive and $1/s^2$ is non negative. Therefore

$$\begin{aligned}
&\mathbb{E}_{q_{k\alpha_T}}\left[\exp\left(\frac{1}{r}\langle M\rangle_T\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\frac{1}{s^2}\langle M\rangle_T + \frac{1}{s}M_T - \frac{1}{s}M_T + \frac{M_T}{k\alpha_T}\right)\right] \\
&\leq \left\{\mathbb{E}\left[\mathcal{E}_T\left(\frac{2}{s}M\right)\right]\right\}^{\frac{1}{2}} \left\{\mathbb{E}\left[\exp\left(\left(-\frac{2}{s} + \frac{2}{k\alpha_T}\right)M_T\right)\right]\right\}^{\frac{1}{2}} \\
&\leq 2,
\end{aligned} \tag{3.21}$$

since

$$-2\left(\frac{1}{s} - \frac{1}{k\alpha_T}\right) = ||M_T||^{-1}_{(\Phi_1,\mu)}. \tag{3.22}$$

Therefore,

$$\mathbb{E}_{q_{k\alpha_T}}\left[\exp\left(\frac{1}{r}\langle M\rangle_T\right)\right] + \mathbb{E}_{q_{k\alpha_T}}\left[\exp\left(-\frac{1}{r}\langle M\rangle_T\right)\right] < 4, \tag{3.23}$$

with strict inequality since $\langle M\rangle_T \neq 0$, so that $||\langle M\rangle_T||_{(\Phi_1,q_{k\alpha_T})} \leq r$. Hence, due to (2.6), the thesis follows immediately for $\tau \equiv T$.

Now, let $\tau \leq T$ such that $M_\tau \neq 0$ and consider $N := M^\tau$; it should be noted that $N \in \mathcal{M}_c$ and $N_T = M_T^\tau = M_\tau \in L^{\Phi_1}(\mu)$ due to (3.5). Hence, (3.18) follows. □

Remark 3. Again, by definition of norm one may obtain the following bound from (3.18)

$$\mathbb{E}\left[\exp\left(\frac{\langle M\rangle_T}{||M_T||^2_{(\Phi_1,\mu)}}\left(\frac{1}{c_k} - \frac{1}{2k^2}\right) + \frac{M_T}{k||M_T||_{(\Phi_1,\mu)}}\right)\right] \leq 4. \tag{3.24}$$

In particular, when $k = 1$, (3.24) reduces to

$$E\left[\exp\left(-\frac{3}{4}\frac{\langle M\rangle_T}{||M_T||^2_{(\Phi_1,\mu)}} + \frac{M_T}{||M_T||_{(\Phi_1,\mu)}}\right)\right] \leq 4. \tag{3.25}$$

Proposition 3.3 and Theorem 3.4 give a BDG-type inequality between the measure μ and a family of measures that depend on the parameter $k \in (2-\sqrt{2}, 2]$. In fact, taking (3.13) with respect to the measure $q_{k\alpha_T}\cdot\mu$ and due to (3.10) and the monotonicity of the norm, the following proposition holds.

Proposition 3.5. *For any non zero $M \in \mathcal{M}_c$, the following holds*

$$\frac{1}{2\sqrt{2}}||M^*||_{(\Phi_1,q_{k\alpha_T}\cdot\mu)} \leq \left\|\langle M\rangle_T^{\frac{1}{2}}\right\|_{(\Phi_2,q_{k\alpha_T}\cdot\mu)} \leq \sqrt{c_k}||M^*||_{(\Phi_1,\mu)}. \tag{3.26}$$

3.2. **Discussion.** It should be noted that $q_{k\alpha_T} \cdot \mu$ actually depends on the considered martingale M. In order to better understand such a structure, it is useful to study the relationships between this class of measures and the reference one μ. For this purpose, we shall prove that, under suitable conditions on M, for each $k \in (1, 2]$, the densities $q_{k\alpha_T}$ and 1 can be connected by a one-dimensional exponential model, so that their corresponding norms are equivalent. Before this, we need the following lemma.

Lemma 3.6. *Let $M \in \mathcal{M}_c$ such that $M_T \in L^{\Phi_1}(\mu)$ and suppose that*

$$1 \le \mathbb{E}_{q_{2\alpha_T}} \left[\cosh\left(r\langle M\rangle_T\right)\right] < \infty \tag{3.27}$$

for some $r > 0$. Then $\langle M\rangle_\tau \in L^{\Phi_1}(\mu)$ for each stopping time $\tau \le T$.

Proof. If $M \equiv 0$, the thesis is trivial; therefore, we can suppose $M_T \neq 0$. Let $p := ||\langle M\rangle_T||_{(\Phi_1, q_{2\alpha_T}\cdot\mu)}$, so that

$$\mathbb{E}_{q_{2\alpha_T}}\left[\exp\left(\frac{\langle M\rangle_T}{p}\right)\right] = \mathbb{E}\left[\exp\left(\frac{M_T}{2\alpha_T} + \frac{\langle M\rangle_T}{p} - \frac{\langle M\rangle_T}{8\alpha_T^2}\right)\right] \le 4 < \infty, \tag{3.28}$$

and define a real positive s in such a way that

$$\frac{4}{s} = \frac{1}{p} - \frac{1}{8\alpha_T^2}. \tag{3.29}$$

In fact, due to the continuity of the function

$$H_u(r) := E\left[\Phi\left(ru\right)\right], \tag{3.30}$$

see e.g. [16, p. 54] condition (3.27) and the strict inequality sign in (3.23) ensure that (3.18) also holds with strict inequality for $k = 2$.

Hence, an application of the generalized Hölder inequality gives

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{\langle M\rangle_T}{s}\right)\right] \\ &\le \left\{\mathbb{E}\left[\exp\left(\frac{-M_T}{2\alpha_T}\right)\right]\right\}^{\frac{1}{4}} \left\{\mathbb{E}\left[\exp\left(\frac{M_T}{2\alpha_T} + \frac{4}{s}\langle M\rangle_T\right)\right]\right\}^{\frac{1}{4}} \\ &\le \left\{\mathbb{E}\left[\exp\left(\frac{-M_T}{2\alpha_T}\right)\right]\right\}^{\frac{1}{4}} \left\{\mathbb{E}\left[\exp\left(\frac{M_T}{2\alpha_T} + \frac{\langle M\rangle_T}{p} - \frac{\langle M\rangle_T}{8\alpha_T^2}\right)\right]\right\}^{\frac{1}{4}} \\ &\le 2 < \infty, \end{aligned} \tag{3.31}$$

due respectively to (2.2) and (3.28). Therefore, there exists $s \in (0, \infty)$ such that

$$\mathbb{E}\left[\exp\left(\frac{\pm\langle M\rangle_T}{s}\right)\right] \le 4 < \infty, \tag{3.32}$$

so that $\langle M\rangle_T \in L^{\Phi_1}(\mu)$. Finally, since the norm is monotone, $\langle M\rangle_\tau \in L^{\Phi_1}(\mu)$ for each $\tau \le T$. □

Remark 4. For instance, condition (3.27) of Lemma 3.6 holds for a continuous martingale $M \in \mathcal{M}_c$ with a bounded quadratic variation.

Proposition 3.7. *Let $M \in \mathcal{M}_c$ be a non zero martingale that satisfies the conditions of Lemma 3.6 and consider $k \in (1, 2]$; then, for each stopping time $\tau \le T$ such that $M_\tau \neq 0$, the two densities 1 and $q_{k\alpha_\tau}$ can be connected by a one-dimensional exponential model. Hence, $||\cdot||_{(\Phi_n, q_{k\alpha_\tau}\cdot\mu)}$ and $||\cdot||_{(\Phi_n,\mu)}$ are equivalent norms.*

Proof. Let $u_\tau := M_\tau/(k\alpha_\tau) - \langle M\rangle_\tau/(2k^2\alpha_\tau^2)$ and define, for an arbitrary small positive ε

$$p(\theta) := \exp(\theta u_\tau - \psi(\theta)), \qquad \theta \in (-\varepsilon, \varepsilon+1), \tag{3.33}$$

where $\psi(\theta) := \log \mathbb{E}[\exp(\theta u_\tau)]$. Due to (3.5) and from Lemma 3.6, $u_\tau \in L^{\Phi_1}(\mu)$; in fact, $p(\theta)$ is an exponential model such that $p(0) = 1$ and $p(1) = q_{k\alpha_\tau}$, the two densities 1 and $q_{k\alpha_\tau}$ being in the interior of the model. Indeed, let us choose $\theta \in (-\varepsilon, 1]$; then

$$\mathbb{E}\left[\mathcal{E}_\tau\left(\frac{M}{k\alpha_\tau}\right)^\theta\right] \leq \mathbb{E}\left[\mathcal{E}_\tau\left(\frac{\theta M}{k\alpha_\tau}\right)\right] \leq 1 < \infty. \tag{3.34}$$

On the other hand, when $\theta \in (1, 1+\varepsilon)$ one obtains

$$\mathbb{E}\left[\mathcal{E}\left(\frac{M_\tau}{k\alpha_\tau}\right)^\theta\right] \leq \mathbb{E}\left[\exp\left(\frac{\theta M_\tau}{k\alpha_\tau}\right)\right] \leq 4 < \infty, \tag{3.35}$$

since $\left|\left|\frac{\theta M_\tau}{k\alpha_\tau}\right|\right|_{(\Phi_1,\mu)} \leq 1$ and due to (2.2). The equivalence of $||\cdot||_{(\Phi_n,\mu)}$ and $||\cdot||_{(\Phi_n,q_{k\alpha_\tau}\cdot\mu)}$ follows from Proposition 2.3. □

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