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A STABILITY OF THE GENERALIZED SINE FUNCTIONAL EQUATIONS, II

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ABSTRACT. The aim of this paper is to study the stability problem of the generalized sine functional equations as follows:

$$g(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2,$$
$$f(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2,$$
$$g(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2.$$

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1. INTRODUCTION

The stability problem of functional equations was raised by S. M. Ulam [11]. Most research follows the Hyers-Ulam stability method which is to construct convergent sequences using an iteration process. In 1979, J. Baker, J. Lawrence and F. Zorzitto in [4] postulated that if f satisfies the stability inequality $|E_1(f) - E_2(f)| \le \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is now frequently referred to as *Superstability*. Baker [3] showed the superstability of the cosine functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$ which is also called the d'Alembert functional equation. The stability of the generalized cosine functional equation has been investigated in many papers ([1], [2], [3], [8], [9]).

The superstability of the generalised sine functional equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2, \tag{S}$$

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has recently been investigated by Cholewa [5], and by Badora and Ger [2].

In this paper, we will introduce the generalized functional equations of the sine equation (S) as follows :

$$(\tilde{S}_{gf}) \qquad g(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2,$$

$$(\tilde{S}_{fg}) \qquad f(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2,$$

$$(\tilde{S}_{gg}) \qquad g(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2.$$

For the case $g = f$, they imply

$$(\tilde{S}) \qquad f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2.$$

Considering the particular case $\sigma(y) = -y$ in the above functional equations, they imply the following functional equations: (S),

$$(S_{gf}) \qquad g(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2,$$

$$(S_{fg}) \qquad f(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2,$$

$$(S_{gg}) \qquad g(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2.$$

Given mappings $f, g : G \to \mathbb{C}$, we define a difference operator $D\tilde{S}_{gf} : G \times G \to \mathbb{C}$ as

$$D\tilde{S}_{gf}(x, y) := g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2.$$

The aim of this paper is to investigate the stability for the generalized sine functional equations ($\tilde{S}_{gf}$), ($\tilde{S}_{fg}$), ($\tilde{S}_{gg}$) under the conditions $|D\tilde{S}_{gf}(x, y)| \le \varepsilon$, $|D\tilde{S}_{fg}(x, y)| \le \varepsilon$, and $|D\tilde{S}_{gg}(x, y)| \le \varepsilon$. From the obtained results, we obtain naturally the stability for the equations (S), ($\tilde{S}$), (S_{gf}), (S_{fg}), (S_{gg}) as corollaries, which can be found in the paper [10].

In this paper, let $(G, +)$ be a uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, $\mathbb{R}$ the field of real numbers, and let σ be an endomorphism of G with $\sigma(\sigma(x)) = x$ for all $x \in G$ with a notation $\sigma(x) = \sigma x$. The properties $g(x) = g(\sigma x)$ and $g(x) = -g(\sigma x)$ with respect to σ will be represented respectively, as even and odd functions for convenience.

We may assume that f and g are nonzero functions and ε is a nonnegative real constant. If all the results of this article are given by the Kannappan condition $f(x + y + z) = f(x + z + y)$ in [7], we will obtain the same results for the semigroup $(G, +)$.

2. STABILITY OF THE EQUATION ($\tilde{S}_{gf}$)

We will investigate the stability of the generalized functional equation ($\tilde{S}_{gf}$) of the sine functional equation (S). From the results we obtain the stability of the functional equations (S), ($\tilde{S}$), (S_{gf}).

Theorem 2.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G. \tag{2.1}$$

Then either g is bounded or f and g are solutions of the functional equation ($\tilde{S}$).

Proof. Let g be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \neq |g(2x_n)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Inequality (2.1) may equivalently be written as

$$|g(2x)f(2y) - f(x+y)^2 + f(x+\sigma y)^2| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.3}$$

Taking $x = x_n$ in (2.3) we obtain

$$\left| f(2y) - \frac{f(x_n+y)^2 - f(x_n+\sigma y)^2}{g(2x_n)} \right| \le \frac{\varepsilon}{|g(2x_n)|},$$

that is, using (2.2)

$$f(2y) = \lim_{n\to\infty} \frac{f(x_n+y)^2 - f(x_n+\sigma y)^2}{g(2x_n)} \qquad \forall\, y \in G. \tag{2.4}$$

Using (2.1) we have

$$\begin{aligned}
2\varepsilon \ge & \left| g(2x_n+x)f(y) - f\left(x_n + \frac{x+y}{2}\right)^2 + f\left(x_n + \frac{x+\sigma y}{2}\right)^2 \right| \\
& + \left| g(2x_n+\sigma x)f(y) - f\left(x_n + \frac{\sigma x+y}{2}\right)^2 + f\left(x_n + \frac{\sigma(x+y)}{2}\right)^2 \right| \\
\ge & \;\Big| \left(g(2x_n+x) + g(2x_n+\sigma x)\right) f(y) \\
& - \left(f\left(x_n + \frac{x+y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+y)}{2}\right)^2 \right) \\
& + \left(f\left(x_n + \frac{x+\sigma y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+\sigma y)}{2}\right)^2 \right) \Big|
\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}
\frac{2\varepsilon}{|g(2x_n)|} \ge & \Bigg| \frac{g(2x_n+x) + g(2x_n+\sigma x)}{g(2x_n)} f(y) \\
& - \frac{f\left(x_n + \frac{x+y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+y)}{2}\right)^2}{g(2x_n)} \\
& + \frac{f\left(x_n + \frac{x+\sigma y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+\sigma y)}{2}\right)^2}{g(2x_n)} \Bigg|
\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Taking the limit as $n \longrightarrow \infty$ with the use of (2.2) and (2.4), we conclude that, for every $x \in G$, there exists the limit

$$h(x) := \lim_{n\to\infty} \frac{g(2x_n+x) + g(2x_n+\sigma x)}{g(2x_n)},$$

where the function $h : G \to \mathbb{C}$ satisfies the equation

$$h(x)f(y) = f(x+y) - f(x+\sigma y) \qquad \forall x, y \in G. \tag{2.5}$$

From the definition of h, we get the equality $h(0) = 2$, which jointly with (2.5) implies that f is an odd function w.r.t. σ, namely, $f(y) = -f(\sigma y)$. Keeping this in mind, by means of (2.5), we infer the equality

$$\begin{aligned} f(x+y)^2 - f(x+\sigma y)^2 &= [f(x+y) + f(x+\sigma y)][f(x+y) - f(x+\sigma y)] \\ &= [f(x+y) + f(x+\sigma y)]h(x)f(y) \\ &= \big[f(2x+y) + f(2x+\sigma y)\big] f(y) \\ &= \big[f(y+2x) - f(y+\sigma(2x))\big] f(y) \\ &= h(y)f(2x)f(y). \end{aligned}$$

The oddness of f forces $f(x+\sigma x) = 0$ for all $x \in G$. Putting $x = y$ in (2.5) we conclude with the above result that

$$f(2y) = f(y)h(y) \qquad \forall\, y \in G.$$

This, in turn, leads to the equation

$$f(x+y)^2 - f(x+\sigma y)^2 = f(2x)f(2y) \qquad \forall\, x, y \in G, \tag{2.6}$$

which, in the light of the unique 2-divisibility of G, gives $(\tilde{S})$.

Next, by showing that $g = f$, we will prove that g is also a solution of $(\tilde{S})$.

If f is bounded, choose $y_0 \in G$ such that $f(2y_0) \neq 0$, and then by (2.3) we obtain

$$\begin{aligned} |g(2x)| - \left| \frac{f(x+y_0)^2 - f(x+\sigma y_0)^2}{f(2y_0)} \right| &\le \left| \frac{f(x+y_0)^2 - f(x+\sigma y_0)^2}{f(2y_0)} - g(2x) \right| \\ &\le \frac{\varepsilon}{|f(2y_0)|} \end{aligned}$$

and it follows that g is also bounded on G.

Since the unbounded assumption of g implies that f is also unbounded, we can choose a sequence $\{y_n\}$ such that $0 \neq |f(2y_n)| \to \infty$ as $n \to \infty$.

A slight change applied after equation (2.2) gives us

$$g(2x) = \lim_{n\to\infty} \frac{f(x+y_n)^2 - f(x+\sigma y_n)^2}{f(2y_n)} \qquad \forall x \in G.$$

Since we have shown that f satisfies (2.6) whenever g is unbounded, the above limit equation is represented as

$$g(2x) = f(2x) \qquad \forall x \in G.$$

By the 2-divisibility of group G, we obtain $f = g$. Therefore we have shown that g also satisfies $(\tilde{S})$. □

Theorem 2.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon, \tag{2.7}$$

which satisfies one of three cases $g(0) = 0$, $g(x) = -g(\sigma x)$, $f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either f is bounded or g satisfies $(\tilde{S})$.

Proof. We use an equivalent equation of (2.7)

$$|g(2x)f(2y) - f(x+y)^2 + f(x+\sigma y)^2| \le \varepsilon \qquad \forall\ x, y \in G. \tag{2.8}$$

Let f be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that

$$0 \ne |f(2y_n)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{2.9}$$

Taking $y = y_n$ in (2.8) we obtain

$$\left| g(2x) - \frac{f(x+y_n)^2 - f(x+\sigma y_n)^2}{f(2y_n)} \right| \le \frac{\varepsilon}{|f(2y_n)|},$$

for all $x \in G$ and all $n \in \mathbb{N}$. This with (2.9) implies that

$$g(2x) = \lim_{n\to\infty} \frac{f(x+y_n)^2 - f(x+\sigma y_n)^2}{f(2y_n)} \quad \text{for all} \quad x \in G. \tag{2.10}$$

An obvious slight change in the steps of the proof applied after formula (2.4) of Theorem 2.1 allows one to state the existence of a limit function

$$h_2(y) := \lim_{n\to\infty} \frac{f(y+2y_n) + f(\sigma y + 2y_n)}{f(2y_n)},$$

where $h_2 : G \to \mathbb{C}$ satisfies the equation

$$g(x)h_2(y) = g(x+y) + g(x+\sigma y) \qquad \forall x, y \in G. \tag{2.11}$$

From the definition of h_2, we have the equality $h_2(y) = h_2(\sigma y)$. Clearly, this applies also to the function $\tilde{h}_2 := \frac{1}{2}h_2$. Moreover, $\tilde{h}_2(0) = \frac{1}{2}h_2(0) = 1$ and

$$g(x+y) + g(x+\sigma y) = 2g(x)\tilde{h}_2(y) \qquad \forall x, y \in G. \tag{2.12}$$

Under (2.12), we know that

$$g(0) = 0 \Longrightarrow g(x) = -g(\sigma x) \Longrightarrow g(x+\sigma x) = 0 \Longrightarrow g(0) = 0. \tag{2.13}$$

Putting $y = x$ in (2.12), we get by (2.13) a duplication formula

$$g(2x) = 2g(x)\tilde{h}_2(x).$$

Using the oddness and duplication of g, we obtain, by means of (2.12), the equation

$$\begin{aligned}
g(x+y)^2 - g(x+\sigma y)^2 &= (g(x+y) + g(x+\sigma y)(g(x+y) - g(x+\sigma y)\\
&= 2g(x)\tilde{h}_2(y)[g(x+y) - g(x+\sigma y)]\\
&= g(x)[g(x+2y) - g(x+2\sigma y)]\\
&= g(x)[g(x+2y) + g(\sigma x+2y)]\\
&= 2g(x)g(2y)\tilde{h}_2(x) = g(2x)g(2y),
\end{aligned}$$

which holds true for all $x, y \in G$, and, in the light of the unique 2-divisibility of G, gives ($\tilde{S}$).

In the last case $f(x)^2 = f(\sigma x)^2$, the proof is completed by showing that $g(0) = 0$. Suppose that this is not the case. Then in what follows, without loss of generality, we may assume that $g(0) = 1$ (replacing, if necessary, the function g by $g/g(0)$ and f by $f/g(0)$).

Putting $x = 0$ in (2.8) with a given condition and the 2-divisibility of group G, we obtain the inequality

$$|f(y)| \le \varepsilon \qquad \forall\ y \in G.$$

This inequality means that f is globally bounded – a contradiction. Thus the claim $g(0) = 0$ holds, so the proof of the theorem is completed. $\square$

By putting $\sigma x = -x$ in Theorems 2.1 and 2.2 respectively, we obtain the following corollaries, respectively.

Corollary 2.3 ([10]). *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon$$

for all $x, y \in G$. Then either g is bounded or f and g are solutions of the equation (S).

Corollary 2.4 ([10]). *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon,$$

which satisfies one of three cases $g(0) = 0$, $g(-x) = -g(x)$, $f(x)^2 = f(-x)^2$ for all $x, y \in G$. Then either f is bounded or g satisfies (S).

Corollary 2.5. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G.$$

Then either f is bounded or f is a solution of the equation ($\tilde{S}$).

Corollary 2.6 ([5]). *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G.$$

Then either f is bounded or f is a solution of the equation (S).

3. STABILITY OF THE EQUATION ($\tilde{S}_{fg}$)

We will investigate the stability of the generalized functional equation ($\tilde{S}_{fg}$) for the sine functional equation (S). The obtained results imply the stability for the functional equations (S), ($\tilde{S}$), (S_{fg}).

Theorem 3.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G. \tag{3.1}$$

Then either f is bounded or g satisfies ($\tilde{S}$).

Proof. Let f be an unbounded solution of the stability inequality (3.1). Then, there exists a sequence $\{x_n\}$ in G such that $0 \ne |f(2x_n)| \to \infty$ as $n \to \infty$.

Putting $x = 2x, y = 2y$ in inequality (3.1), taking $x = x_n$ in the obtained inequality, dividing both sides by $|f(2x_n)|$ and taking the limit as $n \to \infty$ we obtain that

$$g(2y) = \lim_{n\to\infty} \frac{f(x_n + y)^2 - f(x_n + \sigma y)^2}{f(2x_n)} \qquad \forall y \in G. \tag{3.2}$$

An obvious slight change in the steps of the proof applied after (2.4) of Theorem 2.1 in the stability inequality (3.1) allows, with an application of (3.2), us to state the existence of a limit function

$$h_3(x) := \lim_{n\to\infty} \frac{f(2x_n + x) + f(2x_n + \sigma x)}{f(2x_n)},$$

where the function $h_3 : G \to \mathbb{C}$ satisfies the equation

$$h_3(x)g(y) = g(x+y) - g(x+\sigma y) \quad \forall x, y \in G. \tag{3.3}$$

From the definition of h_3, we obtain the equality $h_3(0) = 2$, which with (3.3) implies that g is odd, i.e., $g(y) = -g(\sigma y)$. The oddness of g implies that $g(x + \sigma x) = 0$ for all $x \in G$. Keeping this in mind and putting $x = y$ in (3.3) we conclude that

$$g(2y) = g(y)h_3(y) \quad \text{for all} \quad x, y \in G.$$

Keeping all of these in mind, and by means of (3.3), if we make a slight change of the calculations applied after formula (2.5) of Theorem 2.1, we conclude that the equation

$$g(x+y)^2 - g(x+\sigma y)^2 = g(2x)g(2y)$$

is valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, g gives ($\tilde{S}$). □

Theorem 3.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon, \tag{3.4}$$

which satisfies one of the three cases $f(0) = 0,\ f(x) = -f(\sigma x),\ f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either g is bounded or f satisfies ($\tilde{S}$).

Proof. For an unbounded solution g of the stability inequality (3.4), there exists a sequence $\{y_n\}$ in G such that $0 \neq |g(2y_n)| \to \infty$ as $n \to \infty$.

Following a slight modification in the steps of the proof applied after formula (2.9), we may state the existence of a limit function

$$h_4(y) := \lim_{n\to\infty} \frac{g(y + 2y_n) + g(\sigma y + 2y_n)}{g(2y_n)},$$

where $h_4 : G \to \mathbb{C}$ satisfies the equation

$$f(x)h_4(y) = f(x+y) + f(x+\sigma y) \quad \forall x, y \in G.$$

Using similar proof steps applied after formula (2.11) in Theorem 2.2, we then arrive at the desired result. □

Considering the case $\sigma(x) = -x$ in Theorem 3.1 and Theorem 3.2, we have the following corollaries.

Corollary 3.3 ([10])**.** *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G.$$

Then either f is bounded or g satisfies (S).

Corollary 3.4 ([10])**.** *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon,$$

which satisfies one of the three cases $f(0) = 0, f(x) = -f(-x), f(x)^2 = f(-x)^2$ for all $x, y \in G$. Then either g is bounded or f satisfies (S).

Remark 3.5. Applying $g = f$ in Theorem 3.1 and Theorem 3.2, Corollary 3.3 and Corollary 3.4, we obtain Corollary 2.5 and Corollary 2.6.

4. STABILITY OF THE EQUATION ($\tilde{S}_{gg}$)

We will investigate the stability of the generalized functional equation ($\tilde{S}_{gg}$) of the sine functional equation (S).

Theorem 4.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G. \tag{4.1}$$

Then either g is bounded or g satisfies ($\tilde{S}$).

Proof. Let g be an unbounded solution of the stability inequality (4.1). Then, there exists a sequence $\{x_n\}$ in G such that the relationship (2.2) holds true.

Putting $x = 2x$, $y = 2y$ in inequality (4.1), taking $x = x_n$, dividing both sides by $|g(2x_n)|$ and taking the limit as $n \to \infty$, we obtain that

$$g(2y) = \lim_{n\to\infty} \frac{f(x_n + y)^2 - f(x_n + \sigma y)^2}{g(2x_n)} \tag{4.2}$$

for all $x \in G$.

A slight change in the steps of the proof applied after formula (2.4) in the stability inequalities (4.1) and (4.2), allows one to state the existence of a limit function

$$h_5(x) := \lim_{n\to\infty} \frac{g(2x_n + x) + g(2x_n + \sigma x)}{g(2x_n)},$$

where the function $h_5 : G \to \mathbb{C}$ satisfies the equation

$$h_5(x)g(y) = g(x+y) - g(x+\sigma y) \quad \forall x, y \in G. \tag{4.3}$$

From the definition of h_5, we obtain the equality $h_5(0) = 2$, which with (4.3) implies that $g(y) = -g(\sigma y)$. Keeping this in mind, by means of (4.3), a slight modification applied in the proof after formula (2.5) of Theorem 2.1, gives us the equation

$$g(x+y)^2 - g(x+\sigma y)^2 = g(2x)g(2y)$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, implies ($\tilde{S}$). □

By putting $\sigma x = -x$ or $g = f$ in Theorem 4.1 we obtain the following corollary and the above Corollary 2.5 and Corollary 2.6 as Remark 3.5, respectively.

Corollary 4.2 ([10])**.** *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon \qquad \forall\, x, y \in G.$$

Then either g is bounded or g satisfies (S).

5. APPLICATIONS TO BANACH ALGEBRA

The stability results in Sections 2 to 4 can be extended to Banach algebra. For simplicity, we will combine them according to the following theorems.

Theorem 5.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\left\| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \le \varepsilon \qquad \forall\, x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ g$ is bounded or f and g are solutions of the equation ($\tilde{S}$),*
(ii) *either the superposition $x^* \circ f$ is bounded or ($\tilde{S}$) for g provides us with one of the following cases $g(0) = 0$, $g(x) = -g(\sigma x)$ or $f(x)^2 = f(\sigma x)^2$.*

Proof. The proofs of each case are very similar, so it suffices to show the proof of case (i). Assume that (i) holds and fix an arbitrarily linear multiplicative functional $x^* \in E$. As is well known we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned}
\varepsilon &\geq \left\| g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \\
&= \sup_{\|y^*\|=1} \left| y^*\left(g(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right) \right| \\
&\geq \left| x^*(g(x)) \cdot x^*(f(y)) - x^*\left(f\left(\frac{x+y}{2}\right)^2 \right) + x^*\left(f\left(\frac{x+\sigma y}{2}\right)^2 \right) \right|,
\end{aligned}$$

which states that the superpositions $x^* \circ g$ and $x^* \circ f$ yield a solution of the stability inequality (2.1) of Theorem 2.1. Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2.1 shows that the superpositions $x^* \circ g$ and $x^* \circ f$ solve the generalized sine equation ($\tilde{S}$), respectively. In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the generalized sine difference $\tilde{S}(x, y)$ for the functions f or g falls into the kernel of x^*, respectively. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\tilde{S}(x, y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 = 0 \qquad \forall\, x, y \in G,$$

as claimed, also this is true for g. Case (ii) is similar. □

Since the proofs of the following two theorems also use the same argument as Theorem 5.1, we will omit their proofs for the sake of brevity.

Theorem 5.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\left\| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \leq \varepsilon \qquad \forall\, x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ f$ is bounded or g is a solution of the equation ($\tilde{S}$),*
(ii) *either the superposition $x^* \circ g$ is bounded or f is a solution of the equation ($\tilde{S}$) under one of the cases $g(0) = 0$, $g(x) = -g(\sigma x)$ or $f(x)^2 = f(\sigma x)^2$.*

Theorem 5.3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\left\| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \leq \varepsilon \qquad \forall\, x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ g$ is bounded or g is a solution of the equation ($\tilde{S}$).

Remark 5.4. By applying $\sigma x = -x$ or $g = f$ in Theorem 5.1 to Theorem 5.3, we can obtain the same number of corollaries in Section 2 to Section 4.

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