

ON AN INEQUALITY OF V. CSISZÁR AND T.F. MÓRI FOR CONCAVE FUNCTIONS OF TWO VARIABLES



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Abstract: V. Csiszár and T.F. Móri gave an extension of Diaz-Metcalf's inequality for concave functions. In this paper, we show its restatement. As its applications we first give a reverse inequality of Hölder's inequality. Next we consider two variable versions of Hadamard, Petrović and Giaccardi inequalities.

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and T.F. Móri

Božidar Ivanković, Saichi Izumino,
Josip E. Pečarić and Masaru Tominaga

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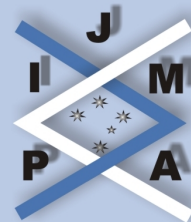
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1. Introduction

In this paper, let (X, Y) be a random vector with $P[(X, Y) \in D] = 1$ where $D := [a, A] \times [b, B]$ ($0 \leq a < A$ and $0 \leq b < B$). Let $E[X]$ be the expectation of a random variable X with respect to P . For a function $\phi : D \rightarrow \mathbb{R}$, we put

$$\Delta\phi = \Delta\phi(a, b, A, B) := \phi(a, b) - \phi(a, B) - \phi(A, b) + \phi(A, B).$$

In [1], V. Csizsár and T.F. Móri showed the following theorem as an extension of Diaz-Metcalf's inequality [2].

Theorem A. *Let $\phi : D \rightarrow \mathbb{R}$ be a concave function.*

We use the following notations:

$$\begin{aligned}\lambda_1 = \lambda_4 &:= \frac{\phi(A, b) - \phi(a, b)}{A - a}, & \mu_1 = \mu_3 &:= \frac{\phi(a, B) - \phi(a, b)}{B - b}, \\ \lambda_2 = \lambda_3 &:= \frac{\phi(A, B) - \phi(a, B)}{A - a}, & \mu_2 = \mu_4 &:= \frac{\phi(A, B) - \phi(A, b)}{B - b}, & \text{and} \\ \nu_1 &:= \frac{AB - ab}{(A - a)(B - b)}\phi(a, b) - \frac{b}{B - b}\phi(a, B) - \frac{a}{A - a}\phi(A, b), \\ \nu_2 &:= \frac{A}{A - a}\phi(a, B) + \frac{B}{B - b}\phi(A, b) - \frac{AB - ab}{(A - a)(B - b)}\phi(A, B), \\ \nu_3 &:= \frac{B}{B - b}\phi(a, b) - \frac{a}{A - a}\phi(A, B) + \frac{aB - Ab}{(A - a)(B - b)}\phi(a, B), \\ \nu_4 &:= \frac{A}{A - a}\phi(a, b) - \frac{b}{B - b}\phi(A, B) - \frac{aB - Ab}{(A - a)(B - b)}\phi(A, b).\end{aligned}$$

a) *Suppose that $\Delta\phi \geq 0$.*

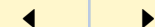
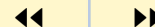
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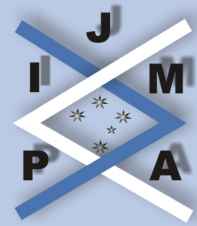
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$a - (i)$ If $(B - b)E[X] + (A - a)E[Y] \leq AB - ab$, then

$$\lambda_1 E[X] + \mu_1 E[Y] + \nu_1 \leq E[\phi(X, Y)] (\leq \phi(E[X], E[Y])).$$

$a - (ii)$ If $(B - b)E[X] + (A - a)E[Y] \geq AB - ab$, then

$$\lambda_2 E[X] + \mu_2 E[Y] + \nu_2 \leq E[\phi(X, Y)] (\leq \phi(E[X], E[Y])).$$

$b)$ Suppose that $\Delta\phi \leq 0$

$b - (iii)$ If $(B - b)E[X] + (A - a)E[Y] \leq aB - Ab$, then

$$\lambda_3 E[X] + \mu_3 E[Y] + \nu_3 \leq E[\phi(X, Y)] (\leq \phi(E[X], E[Y])).$$

$b - (iv)$ If $(B - b)E[X] + (A - a)E[Y] \geq aB - Ab$, then

$$\lambda_4 E[X] + \mu_4 E[Y] + \nu_4 \leq E[\phi(X, Y)] (\leq \phi(E[X], E[Y])).$$

Let us note that Theorem A can be given in the following form:

Theorem 1.1. Suppose that $\phi : D \rightarrow \mathbb{R}$ is a concave function.

$a)$ If $\Delta\phi \geq 0$, then

$$(1.1) \quad \max_{k=1,2} \{\lambda_k E[X] + \mu_k E[Y] + \nu_k\} \leq E[\phi(X, Y)] (\leq \phi(E[X], E[Y])),$$

where λ_k, μ_k and ν_k ($k = 1, 2$) are defined in Theorem A.

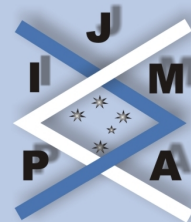
$b)$ If $\Delta\phi \leq 0$, then

$$(1.2) \quad \max_{k=3,4} \{\lambda_k E[X] + \mu_k E[Y] + \nu_k\} \leq E[\phi(X, Y)] (\leq \phi(E[X], E[Y])),$$

where λ_k, μ_k and ν_k ($k = 3, 4$) are defined in Theorem A.

Remark 1. The inequality $E[\phi(X, Y)] \leq \phi(E[X], E[Y])$ is Jensen's inequality. So the inequalities in Theorem A represent reverse inequalities of it.

In this note, we shall give some applications of these results.



2. Reverse Hölder's Inequality

Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\phi(x, y) := x^{\frac{1}{p}}y^{\frac{1}{q}}$ is a concave function on $(0, \infty) \times (0, \infty)$. For $0 < a < A$ and $0 < b < B$, $\Delta\phi$ is represented as follows:

$$\Delta\phi = a^{\frac{1}{p}}b^{\frac{1}{q}} - a^{\frac{1}{p}}B^{\frac{1}{q}} - A^{\frac{1}{p}}b^{\frac{1}{q}} + A^{\frac{1}{p}}B^{\frac{1}{q}} = \left(A^{\frac{1}{p}} - a^{\frac{1}{p}}\right) \left(B^{\frac{1}{q}} - b^{\frac{1}{q}}\right) (> 0).$$

Moreover, putting $A = B = 1$, and replacing X, Y, a and b by X^p, Y^q, α^p and β^q , respectively, in Theorem A, we have the following result:

Theorem 2.1. *Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 < \alpha \leq X \leq 1$ and $0 < \beta \leq Y \leq 1$.*

(i) *If $(1 - \beta^q)E[X^p] + (1 - \alpha^p)E[Y^q] \leq 1 - \alpha^p\beta^q$, then*

$$(2.1) \quad \frac{\beta(1 - \alpha)}{1 - \alpha^p}E[X^p] + \frac{\alpha(1 - \beta)}{1 - \beta^q}E[Y^q] + \frac{\alpha\beta(1 - \alpha^{p-1} - \beta^{q-1} + \alpha^{p-1}\beta^q + \alpha^p\beta^{q-1} - \alpha^p\beta^q)}{(1 - \alpha^p)(1 - \beta^q)} \leq E[XY].$$

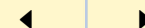
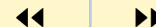
(ii) *If $(1 - \beta^q)E[X^p] + (1 - \alpha^p)E[Y^q] \geq 1 - \alpha^p\beta^q$, then*

$$(2.2) \quad \frac{1 - \alpha}{1 - \alpha^p}E[X^p] + \frac{1 - \beta}{1 - \beta^q}E[Y^q] - \frac{1 - \alpha - \beta + \alpha\beta^q + \alpha^p\beta - \alpha^p\beta^q}{(1 - \alpha^p)(1 - \beta^q)} \leq E[XY].$$

By Theorem 2.1 we have the following inequality related to Hölder's inequality:

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Theorem 2.2. Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. If $0 < \alpha \leq X \leq 1$ and $0 < \beta \leq Y \leq 1$, then

$$(2.3) \quad \begin{aligned} p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha\beta^q)^{\frac{1}{p}} (\alpha - \alpha^p\beta)^{\frac{1}{q}} E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}} \\ \leq (\beta - \alpha\beta^q) E[X^p] + (\alpha - \alpha^p\beta) E[Y^q] \\ \leq (1 - \alpha^p\beta^q) E[XY]. \end{aligned}$$

Proof. We have by Young's inequality

$$\begin{aligned} (\beta - \alpha\beta^q) E[X^p] + (\alpha - \alpha^p\beta) E[Y^q] \\ = \frac{1}{p} \cdot p(\beta - \alpha\beta^q) E[X^p] + \frac{1}{q} \cdot q(\alpha - \alpha^p\beta) E[Y^q] \\ \geq \{p(\beta - \alpha\beta^q) E[X^p]\}^{\frac{1}{p}} \{q(\alpha - \alpha^p\beta) E[Y^q]\}^{\frac{1}{q}} \\ = p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha\beta^q)^{\frac{1}{p}} (\alpha - \alpha^p\beta)^{\frac{1}{q}} E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}}. \end{aligned}$$

Hence the first inequality holds. Next, we see that

$$(2.4) \quad -\gamma_1 := \frac{\alpha\beta(1 - \alpha^{p-1} - \beta^{q-1} + \alpha^{p-1}\beta^q + \alpha^p\beta^{q-1} - \alpha^p\beta^q)}{(1 - \alpha^p)(1 - \beta^q)} \geq 0$$

and

$$(2.5) \quad \gamma_2 := \frac{1 - \alpha - \beta + \alpha\beta^q + \alpha^p\beta - \alpha^p\beta^q}{(1 - \alpha^p)(1 - \beta^q)} \geq 0.$$

Indeed, we have $(1 - \alpha^p)(1 - \beta^q) > 0$ and moreover by Young's inequality

$$\begin{aligned} 1 - \alpha^{p-1} - \beta^{q-1} + \alpha^{p-1}\beta^q + \alpha^p\beta^{q-1} - \alpha^p\beta^q \\ = 1 - \alpha^p\beta^q - \alpha^{p-1}(1 - \beta^q) - \beta^{q-1}(1 - \alpha^p) \\ \geq 1 - \alpha^p\beta^q - \left(\frac{1}{p} + \frac{1}{q}\alpha^p\right)(1 - \beta^q) - \left(\frac{1}{q} + \frac{1}{p}\beta^q\right)(1 - \alpha^p) = 0 \end{aligned}$$

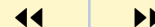
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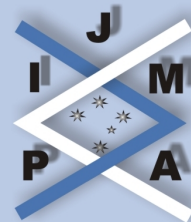
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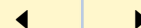
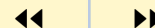
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and

$$\begin{aligned} 1 - \alpha - \beta + \alpha\beta^q + \alpha^p\beta - \alpha^p\beta^q \\ &= 1 - \alpha^p\beta^q - \alpha(1 - \beta^q) - \beta(1 - \alpha^p) \\ &\geq 1 - \alpha^p\beta^q - \left(\frac{1}{q} + \frac{1}{p}\alpha^p\right)(1 - \beta^q) - \left(\frac{1}{p} + \frac{1}{q}\beta^q\right)(1 - \alpha^p) = 0. \end{aligned}$$

Multiplying both sides of (2.1) by γ_2 and those of (2.2) by $-\gamma_1$, respectively, and taking the sum of the two inequalities, we have

$$\left| \begin{array}{cc} \frac{\beta(1-\alpha)}{1-\alpha^p} & \gamma_1 \\ \frac{1-\alpha}{1-\alpha^p} & \gamma_2 \end{array} \right| E[X^p] + \left| \begin{array}{cc} \frac{\alpha(1-\beta)}{1-\beta^q} & \gamma_1 \\ \frac{1-\beta}{1-\beta^q} & \gamma_2 \end{array} \right| E[Y^q] \leq (\gamma_2 - \gamma_1)E[XY].$$

Here we note that from (2.4) and (2.5),

$$\left| \begin{array}{cc} \frac{\beta(1-\alpha)}{1-\alpha^p} & \gamma_1 \\ \frac{1-\alpha}{1-\alpha^p} & \gamma_2 \end{array} \right| = \frac{\beta(1-\alpha)(1-\beta)(1-\alpha\beta^{q-1})}{(1-\alpha^p)(1-\beta^q)},$$

$$\left| \begin{array}{cc} \frac{\alpha(1-\beta)}{1-\beta^q} & \gamma_1 \\ \frac{1-\beta}{1-\beta^q} & \gamma_2 \end{array} \right| = \frac{\alpha(1-\alpha)(1-\beta)(1-\alpha^{p-1}\beta)}{(1-\alpha^p)(1-\beta^q)}$$

and

$$\gamma_2 - \gamma_1 = \frac{(1-\alpha)(1-\beta)(1-\alpha^p\beta^q)}{(1-\alpha^p)(1-\beta^q)}.$$

Hence we have

$$\frac{\beta(1-\alpha)(1-\beta)(1-\alpha\beta^{q-1})}{(1-\alpha^p)(1-\beta^q)} E[X^p] + \frac{\alpha(1-\alpha)(1-\beta)(1-\alpha^{p-1}\beta)}{(1-\alpha^p)(1-\beta^q)} E[Y^q]$$

$$\leq \frac{(1-\alpha)(1-\beta)(1-\alpha^p\beta^q)}{(1-\alpha^p)(1-\beta^q)} E[XY]$$

and so the second inequality of (2.3) holds. \square

The second inequality is given in [5, p.124]. In (2.3), the first and the third terms yield the following Gheorghiu inequality [4, p.184], [5, p.124]:

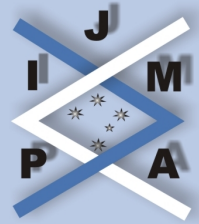
Theorem B. *Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. If $0 < \alpha \leq X \leq 1$ and $0 < \beta \leq Y \leq 1$, then*

$$(2.6) \quad E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}} \leq \frac{1 - \alpha^p \beta^q}{p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha \beta^q)^{\frac{1}{p}} (\alpha - \alpha^p \beta)^{\frac{1}{q}}} E[XY].$$

We see that (2.3) is a kind of a refinement of (2.6). Theorem B gives us the next estimation.

Corollary 2.3. *Let $X = \{a_i\}$ and $Y = \{b_j\}$ be independent discrete random variables with distributions $P(X = a_i) = w_i$ and $P(Y = b_j) = z_j$. Suppose $0 < \alpha \leq X \leq 1$ and $0 < \beta \leq Y \leq 1$. $E[X^p], E[Y^q]$ and $E[XY]$ are given by $\sum_{i=1}^n w_i a_i^p$, $\sum_{j=1}^n z_j b_j^q$ and $\sum_{i=1}^n \sum_{j=1}^n w_i z_j a_i b_j$, respectively. Then we have inequalities*

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n w_i z_j a_i b_j &\leq \left(\sum_{i=1}^n w_i a_i^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n z_j b_j^q \right)^{\frac{1}{q}} \\ &\leq \frac{1 - \alpha^p \beta^q}{p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha \beta^q)^{\frac{1}{p}} (\alpha - \alpha^p \beta)^{\frac{1}{q}}} \sum_{i=1}^n \sum_{j=1}^n w_i z_j a_i b_j. \end{aligned}$$



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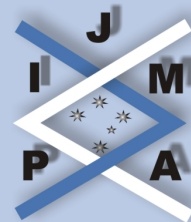
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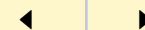
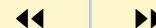
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3. Hadamard's Inequality

The following well-known inequality is due to Hadamard [5, p.11]: For a concave function $f : [a, b] \rightarrow \mathbb{R}$,

$$(3.1) \quad \frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(t) dt \leq f\left(\frac{a+b}{2}\right).$$

Moreover, the following is an extension of the weighted version of Hadamard's inequality by Fejér ([3], [6, p.138]): Let g be a positive integrable function on $[a, b]$ with $g(a+t) = g(b-t)$ for $0 \leq t \leq \frac{1}{2}(a-b)$. Then

$$(3.2) \quad \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \leq \int_a^b f(t)g(t) dt \leq f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt.$$

Here we give an analogous result for a function of two variables.

Theorem 3.1. *Let X and Y be independent random variables such that*

$$(3.3) \quad E[X] = \frac{a+A}{2} \quad \text{and} \quad E[Y] = \frac{b+B}{2}$$

for $0 < a \leq X \leq A$ and $0 < b \leq Y \leq B$. If $\phi : D \rightarrow \mathbb{R}$ is a concave function, then

$$(3.4) \quad \min \left\{ \frac{\phi(A, b) + \phi(a, B)}{2}, \frac{\phi(a, b) + \phi(A, B)}{2} \right\} \leq E[\phi(X, Y)] \\ \leq \phi\left(\frac{a+A}{2}, \frac{b+B}{2}\right).$$

Proof. We only have to prove the case $\Delta\phi \geq 0$. Then with same notations as in Theorem A we have

$$\lambda_1 E[X] + \mu_1 E[Y] + \nu_1 = \lambda_2 E[X] + \mu_2 E[Y] + \nu_2 = \frac{\phi(A, b) + \phi(a, B)}{2}$$



by (3.3). Since $\Delta\phi \geq 0$, it is the same as the first expression in (3.4). Similarly calculation for $\Delta\phi \leq 0$ proves that the desired inequality (3.4) also holds. \square

We can obtain the following result as an extension of Hadamard's inequality (3.1) from Theorem 3.1 by letting X and Y be independent, uniformly distributed random variables on the intervals $[a, A]$ and $[b, B]$, respectively:

Corollary 3.2. *Let $0 < a < A$ and $0 < b < B$. If ϕ is a concave function, then*

$$\min \left\{ \frac{\phi(A, b) + \phi(a, B)}{2}, \frac{\phi(a, b) + \phi(A, B)}{2} \right\} \leq \frac{1}{(A-a)(B-b)} \int_a^A \int_b^B \phi(t, s) ds dt \\ \leq \phi \left(\frac{a+A}{2}, \frac{b+B}{2} \right).$$

By Theorem 3.1, we have the following analogue of (3.2) for a function of two variables:

Corollary 3.3. *Let $w : D \rightarrow \mathbb{R}$ be a nonnegative integrable function such that $w(s, t) = u(s)v(t)$ where $u : [a, A] \rightarrow \mathbb{R}$ is an integrable function with $u(s) = u(a + A - s)$, $\int_a^A u(s) ds = 1$ and $v : [b, B] \rightarrow \mathbb{R}$ is an integrable function such that $\int_b^B v(t) dt = 1$, $v(t) = v(b + B - t)$. If ϕ is a concave function, then*

$$\min \left\{ \frac{\phi(A, b) + \phi(a, B)}{2}, \frac{\phi(a, b) + \phi(A, B)}{2} \right\} \leq \int_a^A \int_b^B w(s, t) \phi(s, t) ds dt \\ \leq \phi \left(\frac{a+A}{2}, \frac{b+B}{2} \right).$$

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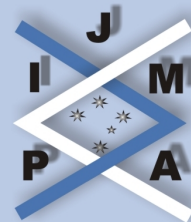
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4. Petrović's Inequality

The following is called Petrović's inequality for a concave function $f : [0, c] \rightarrow \mathbb{R}$:

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) + (1 - P_n) f(0),$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$ are n -tuples of nonnegative real numbers such that $\sum_{i=1}^n p_i x_i \geq x_k$ for $k = 1, \dots, n$, $\sum_{i=1}^n p_i x_i \in [0, c]$ and $P_n := \sum_{i=1}^n p_i$ (see [5, p.11] and [6]).

We give an analogous result for a function of two variables.

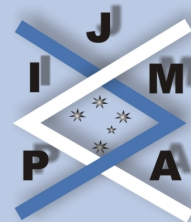
Theorem 4.1. Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be n -tuples of nonnegative real numbers and put $P_n := \sum_{i=1}^n p_i (> 0)$ and $Q_n := \sum_{j=1}^n q_j (> 0)$. Suppose that $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are n -tuples of nonnegative real numbers with $0 \leq x_k \leq \sum_{i=1}^n p_i x_i \leq c$ and $0 \leq y_k \leq \sum_{j=1}^n q_j y_j \leq d$ for $k = 1, 2, \dots, n$. Let $\phi : [0, c] \times [0, d] \rightarrow \mathbb{R}$ be a concave function.

a) Suppose

$$\phi(0, 0) + \phi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) \geq \phi\left(\sum_{i=1}^n p_i x_i, 0\right) + \phi\left(0, \sum_{j=1}^n q_j y_j\right).$$

a - (i) If $\frac{1}{P_n} + \frac{1}{Q_n} \leq 1$, then

$$\begin{aligned} (4.1) \quad \frac{1}{P_n} \phi\left(\sum_{i=1}^n p_i x_i, 0\right) + \frac{1}{Q_n} \phi\left(0, \sum_{j=1}^n q_j y_j\right) + \left(1 - \frac{1}{P_n} - \frac{1}{Q_n}\right) \phi(0, 0) \\ \leq \frac{1}{P_n Q_n} \sum_{i=1}^n \sum_{j=1}^n p_i q_j \phi(x_i, y_j). \end{aligned}$$



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a – (ii) If $\frac{1}{P_n} + \frac{1}{Q_n} \geq 1$, then

$$\begin{aligned} & \left(\frac{1}{P_n} + \frac{1}{Q_n} - 1 \right) \phi \left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j \right) \\ & + \left(1 - \frac{1}{Q_n} \right) \phi \left(\sum_{i=1}^n p_i x_i, 0 \right) + \left(1 - \frac{1}{P_n} \right) \phi \left(0, \sum_{j=1}^n q_j y_j \right) \\ & \leq \frac{1}{P_n Q_n} \sum_{i=1}^n \sum_{j=1}^n p_i q_j \phi(x_i, y_j). \end{aligned}$$

b) Suppose

$$\phi(0, 0) + \phi \left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j \right) \leq \phi \left(\sum_{i=1}^n p_i x_i, 0 \right) + \phi \left(0, \sum_{j=1}^n q_j y_j \right).$$

b – (iii) If $P_n \geq Q_n$, then

$$\begin{aligned} & \frac{1}{P_n} \phi \left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j \right) \\ & + \left(\frac{1}{Q_n} - \frac{1}{P_n} \right) \phi \left(0, \sum_{j=1}^n q_j y_j \right) + \left(1 - \frac{1}{Q_n} \right) \phi(0, 0) \\ & \leq \frac{1}{P_n Q_n} \sum_{i=1}^n \sum_{j=1}^n p_i q_j \phi(x_i, y_j). \end{aligned}$$



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b – (iv) If $Q_n \geq P_n$, then

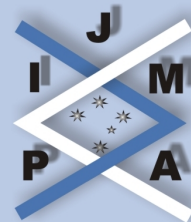
$$\begin{aligned} & \frac{1}{Q_n} \phi \left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j \right) \\ & - \left(\frac{1}{Q_n} - \frac{1}{P_n} \right) \phi \left(\sum_{i=1}^n p_i x_i, 0 \right) + \left(1 - \frac{1}{P_n} \right) \phi(0, 0) \\ & \leq \frac{1}{P_n Q_n} \sum_{i=1}^n \sum_{j=1}^n p_i q_j \phi(x_i, y_j). \end{aligned}$$

Proof. We put $a = b = 0$, $A = \sum_{i=1}^n p_i x_i$ and $B = \sum_{j=1}^n q_j y_j$ in Theorem A. Let $X = \{a_i\}$ and $Y = \{b_j\}$ be independent discrete random variables with distributions $P(X = x_i) = \frac{p_i}{P_n}$ and $P(Y = y_j) = \frac{q_j}{Q_n}$, $1 \leq i \leq n$, respectively. So we have the desired inequalities. \square

Specially, if $p_i = q_j = 1$ ($i, j = 1, \dots, n$) in Theorem 4.1, then we have the following:

Corollary 4.2. Suppose that $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are n -tuples of nonnegative real numbers for $n \geq 2$ with $\sum_{i=1}^n x_i \in [0, c]$ and $\sum_{i=1}^n y_i \in [0, d]$. If $\phi : [0, c] \times [0, d] \rightarrow \mathbb{R}$ is a concave function, then

$$(4.2) \quad \phi \left(\sum_{i=1}^n x_i, 0 \right) + \phi \left(0, \sum_{j=1}^n y_j \right) + (n-2) \phi(0, 0) \leq \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \phi(x_i, y_j).$$



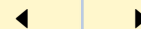
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5. Giaccardi's Inequality

In 1955, Giaccardi (cf. [5, p.11]) proved the following inequality for a convex function $f : [a, A] \rightarrow \mathbb{R}$,

$$\sum_{i=1}^n p_i f(x_i) \leq C \cdot f\left(\sum_{i=1}^n p_i x_i\right) + D \cdot (P_n - 1) \cdot f(x_0),$$

where

$$C = \frac{\sum_{i=1}^n p_i (x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0} \quad \text{and} \quad D = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}$$

for a nonnegative n -tuple $\mathbf{p} = (p_1, \dots, p_n)$ with $P_n := \sum_{i=1}^n p_i$ and a real $(n+1)$ -tuple $\mathbf{x} = (x_0, x_1, \dots, x_n)$ such that for $k = 0, 1, \dots, n$

$$a \leq x_i \leq A, \quad (x_k - x_0) \left(\sum_{i=1}^n p_i x_i - x_0 \right) \geq 0,$$
$$a < \sum_{i=1}^n p_i x_i < A \quad \text{and} \quad \sum_{i=1}^n p_i x_i \neq x_0.$$

In this section, we discuss a generalization of Giaccardi's inequality to a function of two variables under similar conditions. Let $\mathbf{x} = (x_0, x_1, \dots, x_n)$ and $\mathbf{y} = (y_0, y_1, \dots, y_n)$ be nonnegative $(n+1)$ -tuples, and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be nonnegative n -tuples with

$$(5.1) \quad x_0 \leq x_k \leq \sum_{i=1}^n p_i x_i \quad \text{and} \quad y_0 \leq y_k \leq \sum_{j=1}^n q_j y_j \quad \text{for } k = 1, \dots, n.$$



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We use the following notations:

$$P_n := \sum_{i=1}^n p_i (\geq 0),$$

$$Q_n := \sum_{j=1}^n q_j (\geq 0),$$

$$K(X) := \frac{\sum_{i=1}^n p_i x_i - P_n x_0}{\sum_{i=1}^n p_i x_i - x_0},$$

$$K(Y) := \frac{\sum_{j=1}^n q_j y_j - Q_n y_0}{\sum_{j=1}^n q_j y_j - y_0},$$

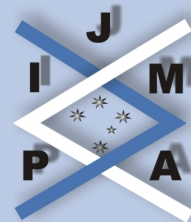
$$L(X) := \frac{(P_n - 1) \sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0},$$

$$L(Y) := \frac{(Q_n - 1) \sum_{j=1}^n q_j y_j}{\sum_{j=1}^n q_j y_j - y_0},$$

$$M(X, Y) := \left\{ (P_n Q_n - P_n - Q_n) \sum_{i=1}^n p_i x_i \sum_{j=1}^n q_j y_j \right. \\ \left. + Q_n y_0 \sum_{i=1}^n p_i x_i + P_n x_0 \sum_{j=1}^n q_j y_j - P_n Q_n x_0 y_0 \right\} \\ \times \frac{1}{\left(\sum_{i=1}^n p_i x_i - x_0 \right) \left(\sum_{j=1}^n q_j y_j - y_0 \right)}$$

and

$$N(X, Y) \\ := \frac{(P_n - Q_n) \sum_{i=1}^n p_i x_i \sum_{j=1}^n q_j y_j - (P_n - 1) Q_n \sum_{i=1}^n p_i x_i y_0 + P_n (Q_n - 1) x_0 \sum_{j=1}^n q_j y_j}{\left(\sum_{i=1}^n p_i x_i - x_0 \right) \left(\sum_{j=1}^n q_j y_j - y_0 \right)}$$



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Then we have the following theorem:

Theorem 5.1. Let $\phi : [x_0, \sum_{i=1}^n p_i x_i] \times [y_0, \sum_{j=1}^n q_j y_j] \rightarrow \mathbb{R}$ be a concave function.

a) If

$$\phi(x_0, y_0) + \phi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) \geq \phi\left(x_0, \sum_{j=1}^n q_j y_j\right) + \phi\left(\sum_{i=1}^n p_i x_i, y_0\right),$$

then

$$\begin{aligned} \max \left\{ Q_n K(X) \phi\left(\sum_{i=1}^n p_i x_i, y_0\right) + P_n K(Y) \phi\left(x_0, \sum_{j=1}^n q_j y_j\right) + M(X, Y) \phi(x_0, y_0), \right. \\ P_n L(Y) \phi\left(\sum_{i=1}^n p_i x_i, y_0\right) + Q_n L(X) \phi\left(x_0, \sum_{j=1}^n q_j y_j\right) \\ \left. - M(X, Y) \phi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) \right\} \leq \sum_{i=1}^n \sum_{j=1}^n p_i q_j \phi(x_i, y_j). \end{aligned}$$

b) If

$$\phi(x_0, y_0) + \phi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) \leq \phi\left(x_0, \sum_{j=1}^n q_j y_j\right) + \phi\left(\sum_{i=1}^n p_i x_i, y_0\right),$$

then

$$\begin{aligned} & \max \left\{ Q_n K(X) \phi \left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j \right) \right. \\ & \quad \left. + P_n L(Y) \phi(x_0, y_0) + N(X, Y) \phi \left(x_0, \sum_{j=1}^n q_j y_j \right), \right. \\ & P_n K(Y) \phi \left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j \right) + Q_n L(X) \phi(x_0, y_0) \\ & \quad \left. - N(X, Y) \phi \left(\sum_{i=1}^n p_i x_i, y_0 \right) \right\} \leq \sum_{i=1}^n \sum_{j=1}^n p_i q_j \phi(x_i, y_j). \end{aligned}$$

Proof. Let X and Y be as they were in the proof of Theorem 4.1, and put $a = x_0$, $A = \sum_{i=1}^n p_i x_i$, $b = y_0$ and $B = \sum_{j=1}^n q_j y_j$, and use Theorem A. Then we have the desired inequalities of this theorem. \square



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