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HERMITE-HADAMARD TYPE INEQUALITIES FOR INCREASING RADIANT FUNCTIONS

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Abstract

We study Hermite-Hadamard type inequalities for increasing radiant functions and give some simple examples of such inequalities.

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Contents

1. Introduction

In this paper we consider one generalization of Hermite-Hadamard inequalities for the class InR of increasing radiant functions defined on the cone $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0 \ (i = 1, \ldots, n)\}$.

Recall that for a function $f : [a, b] \to \mathbb{R}$, which is convex on $[a, b]$, we have the following:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{2}(f(a)+f(b)). \tag{1.1}$$

These inequalities are well known as the Hermite-Hadamard inequalities. There are many generalizations of these inequalities for classes of non-convex functions. For more information see ([2], Section 6.5), [1] and references therein. In this paper we consider generalizations of the inequalities from both sides of (1.1). Some techniques and notions, which are used here, can be found in [1].

In Section 2 of this paper we give a definition of InR functions and recall some results related to these functions. In Section 3 we consider Hermite-Hadamard type inequalities for the class InR. Some examples of such inequalities for functions defined on $\mathbb{R}_{++}$ and $\mathbb{R}^2_{++}$ are given in Section 4.

2. Preliminaries

We assume that the cone $\mathbb{R}^n_{++}$ is equipped with coordinate-wise order relation.

Recall that a function $f : \mathbb{R}^n_{++} \to \bar{\mathbb{R}}_+ = [0, +\infty]$ is said to be increasing radiant (InR) if:

1. f is increasing: $x \geq y \implies f(x) \geq f(y)$;
2. f is radiant: $f(\lambda x) \leq \lambda f(x)$ for all $\lambda \in (0,1)$ and $x \in \mathbb{R}^n_{++}$.

For example, any function f of the following form belongs to the class InR:

$$f(x) = \sum_{|k| \geq 1} c_k x_1^{k_1} \cdots x_n^{k_n},$$

where $k = (k_1, \dots, k_n)$, $|k| = k_1 + \cdots + k_n$, $k_i \geq 0$, $c_k \geq 0$.

For each $f \in InR$ its conjugate function ([4])

$$f^*(x) = \frac{1}{f(1/x)},$$

where $1/x = (1/x_1, \dots, 1/x_n)$, is also increasing and radiant. Hence any function

$$f(x) = \frac{1}{\sum_{|k| \geq 1} c_k x_1^{-k_1} \cdots x_n^{-k_n}}$$

is InR. In the more general case we have the following InR functions:

$$f(x) = \left(\frac{\sum_{|k| \geq u} c_k x_1^{k_1} \cdots x_n^{k_n}}{\sum_{|k| \geq v} d_k x_1^{-k_1} \cdots x_n^{-k_n}} \right)^t,$$

where $u, v > 0$, $t \geq 1/(u+v)$. Indeed, these functions are increasing and for any $\lambda \in (0,1)$

$$\begin{aligned} f(\lambda x) &= \left(\frac{\sum_{|k|\geq u} \lambda^{|k|} c_k x_1^{k_1} \cdots x_n^{k_n}}{\sum_{|k|\geq v} \lambda^{-|k|} d_k x_1^{-k_1} \cdots x_n^{-k_n}} \right)^t \\ &\leq \left(\frac{\lambda^u \sum_{|k|\geq u} c_k x_1^{k_1} \cdots x_n^{k_n}}{\lambda^{-v} \sum_{|k|\geq v} d_k x_1^{-k_1} \cdots x_n^{-k_n}} \right)^t \\ &= \lambda^{(u+v)t} f(x) \leq \lambda f(x). \end{aligned}$$

Consider the coupling function φ defined on $\mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$:

$$\varphi(h,x) = \begin{cases} 0, & \text{if } \langle h, x \rangle < 1, \\ \langle h, x \rangle, & \text{if } \langle h, x \rangle \geq 1, \end{cases} \tag{2.1}$$

where

$$\langle h, x \rangle = \min\{h_i x_i : i = 1, \dots, n\}$$

is the so-called min-type function.

Denote by φ_h the function defined on $\mathbb{R}^n_{++}$ by the formula: $\varphi_h(x) = \varphi(h, x)$.

It is known (see [4]) that the set

$$H = \left\{ \frac{1}{c} \varphi_h : h \in \mathbb{R}^n_{++},\ c \in (0, +\infty] \right\}$$

is the supremal generator of the class InR of all increasing radiant functions defined on $\mathbb{R}^n_{++}$.

It is known also that for any InR function f

$$f(h)\varphi\left(\frac{1}{h}, x\right) \le f(x) \quad \text{for all } x, h \in \mathbb{R}^n_{++}. \tag{2.2}$$

Note that for $c = +\infty$ we set $c\varphi_h(x) = \sup_{l>0}(l\varphi_h(x))$.

Formula (2.2) implies the following statement.

Proposition 2.1. *Let f be an InR function defined on $\mathbb{R}^n_{++}$ and $\Delta \subset \mathbb{R}^n_{++}$. Then the function*

$$f_\Delta(x) = \sup_{h\in\Delta} f(h)\varphi\left(\frac{1}{h}, x\right)$$

is InR, and it possesses the properties:

1) $f_\Delta(x) \le f(x)$ *for all* $x \in \mathbb{R}^n_{++}$,
2) $f_\Delta(x) = f(x)$ *for all* $x \in \Delta$.

3. Hermite-Hadamard Type Inequalities

Let $D \subset \mathbb{R}^n_{++}$ be a closed domain (in topology of $\mathbb{R}^n_{++}$), i.e. D is a bounded set such that $\operatorname{cl}\operatorname{int} D = D$. Denote by $Q(D)$ the set of all points $\bar{x} \in D$ such that

$$\frac{1}{A(D)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1, \tag{3.1}$$

where $A(D) = \int_D dx, \quad dx = dx_1 \cdots dx_n$.

Proposition 3.1. *Let f be an InR function defined on $\mathbb{R}^n_{++}$. If the set $Q(D)$ is nonempty and f is integrable on D then*

$$\sup_{\bar{x}\in Q(D)} f(\bar{x}) \le \frac{1}{A(D)} \int_D f(x)dx. \tag{3.2}$$

Proof. First, let $\bar{x} \in Q(D)$ and $f(\bar{x}) < +\infty$. Then $f(\bar{x})\varphi(1/\bar{x}, x) \le f(x)$ for all $x \in D \subset \mathbb{R}^n_{++}$ (see (2.2)). By (3.1), we get

$$\begin{aligned} f(\bar{x}) &= f(\bar{x})\frac{1}{A(D)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx \\ &= \frac{1}{A(D)} \int_D f(\bar{x})\varphi\left(\frac{1}{\bar{x}}, x\right) dx \\ &\le \frac{1}{A(D)} \int_D f(x)dx. \end{aligned}$$

Now, suppose that $f(\bar{x}) = +\infty$. Then for all $l > 0$ function $l\varphi_{1/\bar{x}}(x)$ is minorant of f. Hence $l \le \frac{1}{A(D)} \int_D f(x)\,dx \quad \forall\, l > 0$, that implies that function f is not integrable on D. This contradiction shows that $f(\bar{x}) < +\infty$ for any $\bar{x} \in Q(D)$. □

As it was done in [1], we may introduce the set $Q_m(D)$ of all maximal elements of $Q(D)$. It means that a point $\bar{x} \in Q(D)$ belongs to $Q_m(D)$ if and only if for any $\bar{y} \in Q(D) : (\bar{y} \geq \bar{x}) \implies (\bar{y} = \bar{x})$. Suppose that the set $Q(D)$ is nonempty. It is easy to see that $Q(D)$ is a closed set in the topology of $\mathbb{R}^n_{++}$. Hence, using the Zorn Lemma we conclude that $Q_m(D)$ is a nonempty closed set and for any $\bar{x} \in Q(D)$ there exists $\bar{y} \in Q_m(D)$, for which $\bar{x} \leq \bar{y}$.

So, in assumptions of Proposition 3.1 we have the following estimate:

$$\sup_{\bar{x} \in Q_m(D)} f(\bar{x}) \leq \frac{1}{A(D)} \int_D f(x)dx. \tag{3.3}$$

Since f is an increasing function then this inequality implies inequality (3.2).

Remark 3.1. *Let $D \subset \mathbb{R}^n_{++}$ be a closed domain and the set $Q(D)$ be nonempty. Then for every $\bar{x} \in Q(D)$ inequality*

$$f(\bar{x}) \leq \frac{1}{A(D)} \int_D f(x)dx$$

is sharp. For example, if we set $f = \varphi_{1/\bar{x}}$ then (see (3.1))

$$f(\bar{x}) = \varphi\left(\frac{1}{\bar{x}}, \bar{x}\right) = 1 = \frac{1}{A(D)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \frac{1}{A(D)} \int_D f(x)dx.$$

Note that here we used only the values of function f on a set D. Therefore we need the following definition.

Definition 3.1. *Let $D \subset \mathbb{R}^n_{++}$. A function $f : D \to [0, +\infty]$ is said to be increasing radiant on D if there exists an InR function F defined on $\mathbb{R}^n_{++}$ such that $F|_D = f$, that is $F(x) = f(x)$ for all $x \in D$.*

We assume here, as above, that for $c = +\infty : c\varphi_h(x) = \sup_{l>0}(l\varphi_h(x))$.

Proposition 3.2. *Let* $f : D \to [0, +\infty]$ *be a function defined on* $D \subset \mathbb{R}^n_{++}$. *Then the following assertions are equivalent:*

1) f *is increasing radiant on* D,

2) $f(h)\varphi(1/h, x) \leq f(x)$ *for all* $h, x \in D$,

3) f *is abstract convex with respect to the set of functions* $(1/c)\varphi_{(1/h)} : D \to [0, +\infty]$ *with* $h \in D$, $c \in (0, +\infty]$.

Proof. 1)$\Longrightarrow$2). By Definition 3.1, there exists an InR function $F : \mathbb{R}^n_{++} \to [0, +\infty]$ such that $F(x) = f(x)$ for all $x \in D$. Then Proposition 2.1 implies that the function

$$F_D(x) = \sup_{h\in D} F(h)\varphi\left(\frac{1}{h}, x\right)$$

interpolates F in all points $x \in D$. Hence

$$\sup_{h\in D} f(h)\varphi\left(\frac{1}{h}, x\right) = f(x) \text{ for all } x \in D,$$

that implies the assertion 2)

2)$\Longrightarrow$3). Consider the function f_D defined on D

$$f_D(x) = \sup_{h\in D} f(h)\varphi\left(\frac{1}{h}, x\right).$$

First, it is clear that f_D is abstract convex with respect to the set of functions defined on D : $\{(1/c)\varphi_{(1/h)} : h \in D,\ \ c \in (0, +\infty]\}$. Further, using 2) we get for all $x \in D$

$$f_D(x) \le f(x) = f(x)\varphi\left(\frac{1}{x}, x\right) \le \sup_{h\in D} f(h)\varphi\left(\frac{1}{h}, x\right) = f_D(x).$$

So, $f_D(x) = f(x)$ for all $x \in D$ and we have the desired statement 3).

3)$\Longrightarrow$1). It is obvious since any function $(1/c)\varphi_h$ defined on D can be considered as an elementary function $(1/c)\varphi_h \in H$ defined on $\mathbb{R}^n_{++}$. □

Remark 3.2. *We may require in Proposition 3.1, formula (3.3) and Remark 3.1 only that function f is increasing radiant and integrable on D.*

Remark 3.3. *We may consider a more general case of Hermite-Hadamard type inequalities for InR functions. Let f be an increasing radiant function on D. Then Proposition 3.2 implies that $f(h)\varphi(1/h, x) \le f(x)$ for all $h, x \in D$. If $f(\bar{x}) < +\infty$ and f is integrable on D then*

$$f(\bar{x}) \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx \le \int_D f(x)dx. \tag{3.4}$$

This inequality is sharp for any $\bar{x} \in D$ since we have the equality in (3.4) for $f = \varphi_{(1/\bar{x})}$.

Proposition 3.2 implies also that the class InR is broad enough.

Proposition 3.3. *Let $S \subset \mathbb{R}^n_{++}$ be a set such that every point $x \in S$ is maximal in S. Then for any function $f : S \to [0, +\infty]$ there exists an increasing radiant function $F : \mathbb{R}^n_{++} \to [0, +\infty]$, for which $F|_S = f$.*

Proof. It is sufficient to check only that $f(h)\varphi(1/h,x) \le f(x)$ for all $h,x \in S$. If $h = x$ then $\varphi(1/h,x) = 1$, $f(h) = f(x)$. If $h \ne x$ then $\langle 1/h, x\rangle = \min_i x_i/h_i < 1$ since h is a maximal point in S, hence $\varphi(1/h,x) = 0$ and $f(h)\varphi(1/h,x) = 0 \le f(x)$. $\square$

In particular, Proposition 3.3 holds if $S = \{x \in \mathbb{R}^n_{++} : (x_1)^p + \cdots + (x_n)^p = 1\}$, where $p > 0$.

Now we present two assertions supported by the definition of function φ. Recall that a set $\Omega \subset \mathbb{R}^n_{++}$ is said to be normal if for each $x \in \Omega$ we have $(y \in \Omega$ for all $y \le x)$. The *normal hull* $N(\Omega)$ of a set Ω is defined as follows: $N(\Omega) = \{x \in \mathbb{R}^n_{++} : (\exists\, y \in \Omega)\ x \le y\}$ (see, for example, [3]).

Proposition 3.4. *Let $D, \Omega \subset \mathbb{R}^n_{++}$ be closed domains and $D \subset \Omega$. If the set $Q(\Omega)$ is nonempty and*

$$(\Omega \backslash D) \subset N(Q(\Omega)) \tag{3.5}$$

then the set $Q(D)$ consists of all points $\bar{x} \in \Omega$ such that

$$\frac{1}{A(D)} \int_\Omega \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1.$$

Proof. If $D = \Omega$ then the assertion is clear. Assume that $D \ne \Omega$. Since D, Ω are closed domains and $D \subset \Omega$ then

$$A(D) < A(\Omega). \tag{3.6}$$

Let $\bar{x} \in \Omega$ and

$$\frac{1}{A(D)} \int_\Omega \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1. \tag{3.7}$$

We show that $\varphi(1/\bar{x}, x) = 0$ for all $x \in \Omega \backslash D$. If $x \in \Omega \backslash D$ then, by (3.5), there exists a point $\bar{y} \in Q(\Omega) : \bar{y} \geq x$; hence $\langle 1/\bar{x}, x\rangle \leq \langle 1/\bar{x}, \bar{y}\rangle$. Suppose that $\langle 1/\bar{x}, \bar{y}\rangle \geq 1$. Then $\bar{y} \geq \bar{x} \Longrightarrow 1/\bar{y} \leq 1/\bar{x}$. Since $\bar{y} \in Q(\Omega)$ then, by (3.6) and (3.7)

$$\begin{aligned} 1 &= \frac{1}{A(\Omega)} \int_{\Omega} \varphi\left(\frac{1}{\bar{y}}, x\right) dx \\ &< \frac{1}{A(D)} \int_{\Omega} \varphi\left(\frac{1}{\bar{y}}, x\right) dx \\ &\leq \frac{1}{A(D)} \int_{\Omega} \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1. \end{aligned}$$

So, we have the inequalities: $\langle 1/\bar{x}, x\rangle \leq \langle 1/\bar{x}, \bar{y}\rangle < 1$. Therefore $\varphi(1/\bar{x}, x) = 0$ for all $x \in \Omega \backslash D \Longrightarrow$

$$1 = \frac{1}{A(D)} \int_{\Omega} \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \frac{1}{A(D)} \int_{D} \varphi\left(\frac{1}{\bar{x}}, x\right) dx.$$

The equality ($\varphi(1/\bar{x}, \cdot) = 0$ on $\Omega \backslash D$) implies also that $\bar{x} \neq x$ for all $x \in \Omega \backslash D$, hence $\bar{x} \notin \Omega \backslash D \Longrightarrow \bar{x} \in D$. Thus, we have the established result: $\bar{x} \in Q(D)$.

Conversely, let $\bar{x} \in Q(D)$. For any $x \in \Omega \backslash D$ there exists $\bar{y} \in Q(\Omega)$ such that $\bar{y} \geq x \Longrightarrow \langle 1/\bar{x}, x\rangle \leq \langle 1/\bar{x}, \bar{y}\rangle$. Moreover, we may assume that $\bar{y}$ is a maximal point in $Q(\Omega)$, i.e. $\bar{y} \in Q_m(\Omega)$. First, we check that

$$\left\langle \frac{1}{\bar{y}}, x \right\rangle \leq 1 \text{ for all } x \in \Omega \backslash D,\ \bar{y} \in Q_m(\Omega). \tag{3.8}$$

Indeed, if $x \in \Omega \backslash D$ then for some $\bar{z} \in Q_m(\Omega)$: $x \le \bar{z} \Longrightarrow \langle 1/\bar{y}, x\rangle \le \langle 1/\bar{y}, \bar{z}\rangle$. But $\langle 1/\bar{y}, \bar{z}\rangle \le 1$ since $\bar{y}, \bar{z} \in Q_m(\Omega)$ (otherwise, if $\langle 1/\bar{y}, \bar{z}\rangle > 1$ then $\bar{z} > \bar{y} \Longrightarrow \bar{y} \notin Q_m(\Omega)$).

Now we verify that $\langle 1/\bar{x}, x\rangle < 1$ for all $x \in \Omega \backslash D$. If $x \in \Omega \backslash D$ then for some $\bar{y} \in Q_m(\Omega) : \langle 1/\bar{x}, x\rangle \le \langle 1/\bar{x}, \bar{y}\rangle$. Suppose that $\langle 1/\bar{x}, \bar{y}\rangle \ge 1$. Then $\bar{y} \ge \bar{x}$ and therefore, using inclusion $\bar{x} \in Q(D)$, we get

$$1 = \frac{1}{A(D)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx > \frac{1}{A(\Omega)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx \ge \frac{1}{A(\Omega)} \int_D \varphi\left(\frac{1}{\bar{y}}, x\right) dx. \tag{3.9}$$

Let $D_1 = \{x \in \Omega \backslash D : \langle 1/\bar{y}, x\rangle < 1\}$, $D_2 = \{x \in \Omega \backslash D : \langle 1/\bar{y}, x\rangle = 1\}$. It follows from (3.8) that $\Omega \backslash D = D_1 \cup D_2$ ($D_1 \cap D_2 = \emptyset$), hence

$$\int_{\Omega \backslash D} \varphi\left(\frac{1}{\bar{y}}, x\right) dx = \int_{D_1} \varphi\left(\frac{1}{\bar{y}}, x\right) dx + \int_{D_2} \varphi\left(\frac{1}{\bar{y}}, x\right) dx = \int_{D_2} \varphi\left(\frac{1}{\bar{y}}, x\right) dx = \int_{D_2} dx.$$

But the last integral $\int_{D_2} dx$ is also equal to zero, since the set D_2 has no interior points. Thus, by (3.9)

$$1 > \frac{1}{A(\Omega)} \int_D \varphi\left(\frac{1}{\bar{y}}, x\right) dx = \frac{1}{A(\Omega)} \int_\Omega \varphi\left(\frac{1}{\bar{y}}, x\right) dx.$$

This inequality contradicts the inclusion $\bar{y} \in Q_m(\Omega)$. So, we conclude that the inequality $\langle 1/\bar{x}, \bar{y} \rangle \geq 1$ is impossible. Hence $\langle 1/\bar{x}, x \rangle \leq \langle 1/\bar{x}, \bar{y} \rangle < 1$ for all $x \in \Omega \backslash D$ and $\bar{y} = \bar{y}(x) \in Q_m(\Omega)$, which implies the required equality:

$$1 = \frac{1}{A(D)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \frac{1}{A(D)} \int_\Omega \varphi\left(\frac{1}{\bar{x}}, x\right) dx.$$

□

Corollary 3.5. *Let $D_1, D_2 \subset \mathbb{R}^n_{++}$ be a closed domains such that*

$$A(D_1) = A(D_2).$$

If there exists a closed domain $\Omega \subset \mathbb{R}^n_{++}$, for which the set $Q(\Omega)$ is nonempty and

$$D_i \subset \Omega, \quad (\Omega \backslash D_i) \subset N(Q(\Omega)) \qquad (i = 1, 2),$$

then

$$Q(D_1) = Q(D_2).$$

Proposition 3.6. *Let $D, \Omega \subset \mathbb{R}^n_{++}$ be closed domains and $D \subset \Omega$. If*

$$N(\Omega \backslash D) \cap D = \emptyset, \tag{3.10}$$

then the set $Q(D)$ consists of all points $\bar{x} \in D$ such that

$$\frac{1}{A(D)} \int_\Omega \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1.$$

Proof. Formula (3.10) implies that if $\bar{x} \in D$ then $\bar{x} \notin N(\Omega \setminus D)$. It means that for all

$$x \in \Omega \setminus D : x < \bar{x} \Longrightarrow \left\langle \frac{1}{\bar{x}}, x \right\rangle < 1 \Longrightarrow \varphi\left(\frac{1}{\bar{x}}, x\right) = 0.$$

Thus, for any $\bar{x} \in D$

$$\frac{1}{A(D)} \int_{\Omega} \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1 \Longleftrightarrow \frac{1}{A(D)} \int_{D} \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1$$
$$\Longleftrightarrow \bar{x} \in Q(D).$$

□

Now consider the generalization of the inequality from the right-hand side of (1.1). Let f be an increasing radiant function defined on a closed domain $D \subset \mathbb{R}^n_{++}$, and f is integrable on D. Then $f(h)\varphi(1/h, x) \leq f(x)$ for all $h, x \in D$. In particular, $f(h)\langle 1/h, x\rangle \leq f(x)$ if $\langle 1/h, x\rangle \geq 1$. Hence for all $x \geq h$

$$f(h) \leq \frac{f(x)}{\langle 1/h, x\rangle} = \left\langle h, \frac{1}{x} \right\rangle^{+} f(x),$$

where $h(y) = \langle h, y\rangle^{+} = \max_i h_i y_i$ is the so-called max-type function. So, if $\bar{x} \in D$ and $\bar{x} \geq x$ for all $x \in D$, then $f(x) \leq \langle x, 1/\bar{x}\rangle^{+} f(\bar{x})$ for any $\bar{x} \in D$. This reduces to the following assertion.

Proposition 3.7. *Let the function f be increasing radiant and integrable on D. If $\bar{x} \in D$ and $\bar{x} \geq x$ for all $x \in D$, then*

$$\int_D f(x)dx \leq f(\bar{x}) \int_D \left\langle x, \frac{1}{\bar{x}} \right\rangle^{+} dx. \tag{3.11}$$

Inequality (3.11) is sharp since we get equality for $f(x) = \langle x, 1/\bar{x}\rangle^{+}$.

In the more general case we have the following inequalities:

$$f(x) \le \langle x, 1/\bar{x}\rangle^{+} \sup_{y\in D} f(y) \quad \text{for all } \bar{x} \ge x.$$

Hence

$$f(x) \le \sup_{y\in D} f(y) \inf \left\{ \left\langle x, \frac{1}{\bar{x}} \right\rangle^{+} : \bar{x} \ge x,\ \bar{x} \in D \right\} \quad \text{for all } x \in D$$

and therefore

$$\int_D f(x)dx \le \sup_{y\in D} f(y) \int_D \inf \left\{ \left\langle x, \frac{1}{\bar{x}} \right\rangle^{+} : \bar{x} \ge x,\ \bar{x} \in D \right\} dx. \tag{3.12}$$

4. Examples

Here we describe the set $Q(D)$ for some special domains D of the cones $\mathbb{R}_{++}$ and $\mathbb{R}^2_{++}$.

Let $a, b \in \mathbb{R}$ be numbers such that $0 \leq a < b$. We denote by $[a, b]$ the segment $\{x \in \mathbb{R}_{++} : a \leq x \leq b\}$.

Example 4.1. *Let $D = [a, b] \subset \mathbb{R}_{++}$, where $0 \leq a < b$. By definition, the set $Q(D)$ consists of all points $\bar{x} \in D$, for which*

$$\frac{1}{A(D)} \int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \frac{1}{b-a} \int_a^b \varphi\left(\frac{1}{\bar{x}}, x\right) dx = 1.$$

We have:

$$\varphi\left(\frac{1}{\bar{x}}, x\right) = \begin{cases} 0, & \text{if } x < \bar{x}, \\ \frac{x}{\bar{x}}, & \text{if } x \geq \bar{x}. \end{cases}$$

Hence, if $\bar{x} \in D = [a, b]$ then

$$\int_a^b \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \int_{\bar{x}}^b \frac{x}{\bar{x}} dx = \frac{1}{2\bar{x}}(b^2 - \bar{x}^2). \tag{4.1}$$

So, a point $\bar{x} \in [a, b]$ belongs to $Q(D)$ if and only if

$$\frac{1}{2(b-a)\bar{x}}(b^2 - \bar{x}^2) = 1 \iff \bar{x}^2 + 2(b-a)\bar{x} - b^2 = 0.$$

We get

$$\bar{x} = \sqrt{(b-a)^2 + b^2} - (b-a). \tag{4.2}$$

Show that for the point (4.2)

$$a < \bar{x} < \frac{a+b}{2}. \tag{4.3}$$

Since $b > a \geq 0$ *then* $\bar{x} = \sqrt{(b-a)^2 + b^2} - (b-a) > \sqrt{b^2} - (b-a) = a$. *Further,*

$$\begin{aligned} \bar{x} < \frac{a+b}{2} &\iff \sqrt{(b-a)^2 + b^2} < (b-a) + \frac{a+b}{2} = \frac{3b-a}{2} \\ &\iff 4(b-a)^2 + 4b^2 < (3b-a)^2 \\ &\iff 0 < b^2 + 2ab - 3a^2. \end{aligned}$$

The last inequality follows from the same conditions $b > a \geq 0$.

Thus, $Q([a,b]) = \left\{\sqrt{(b-a)^2 + b^2} - (b-a)\right\}$. *Remark 3.1 implies that for every* InR *function* $f \in L_1[a,b]$

$$f\left(\sqrt{(b-a)^2 + b^2} - (b-a)\right) \leq \frac{1}{b-a}\int_a^b f(x)dx$$

and this inequality is sharp. (Compare it with the corresponding estimate for convex functions (1.1), see also (4.3)).

Remark 3.3 and formula (4.1) imply the following inequalities

$$f(u) \leq \frac{2u}{b^2 - u^2}\int_a^b f(x)dx, \tag{4.4}$$

which are sharp in the class of all InR functions $f \in L_1[a,b]$ and hold for any $u \in [a,b)$. In particular, we get for $u = (a+b)/2$

$$f\left(\frac{a+b}{2}\right) \leq \frac{4(a+b)}{(a+3b)(b-a)} \int_a^b f(x)dx.$$

Note that here

$$\frac{4(a+b)}{(a+3b)(b-a)} > \frac{1}{b-a}.$$

Further, Proposition 3.7 implies that

$$\int_a^b f(x)dx \leq f(b) \int_a^b \frac{x}{b}dx = \frac{b^2-a^2}{2b} f(b),$$

hence

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{a+b}{2b} f(b)$$

for every InR function $f \in L_1[a,b]$.

Let $D \subset \mathbb{R}^2_{++}$, $\bar{x} = (\bar{x}_1, \bar{x}_2) \in D$. We denote by $D(\bar{x})$ the set $\{x \in D : x_1 \geq \bar{x}_1,\ x_2 \geq \bar{x}_2\}$. It is clear that

$$\int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \int_{D(\bar{x})} \left\langle \frac{1}{\bar{x}}, x \right\rangle dx = \int_{D(\bar{x})} \min\left(\frac{x_1}{\bar{x}_1}, \frac{x_2}{\bar{x}_2}\right) dx_1 dx_2.$$

In order to calculate such integrals we represent the set $D(\bar{x})$ as a union $D_1(\bar{x}) \cup D_2(\bar{x})$, where

$$D_1(\bar{x}) = \left\{x \in D(\bar{x}) : \frac{x_2}{\bar{x}_2} \leq \frac{x_1}{\bar{x}_1}\right\}, \quad D_2(\bar{x}) = \left\{x \in D(\bar{x}) : \frac{x_1}{\bar{x}_1} \leq \frac{x_2}{\bar{x}_2}\right\}.$$

Then

$$\int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \int_{D_1(\bar{x})} \left\langle \frac{1}{\bar{x}}, x \right\rangle dx + \int_{D_2(\bar{x})} \left\langle \frac{1}{\bar{x}}, x \right\rangle dx$$
$$= \frac{1}{\bar{x}_2} \int_{D_1(\bar{x})} x_2 \, dx_1 dx_2 + \frac{1}{\bar{x}_1} \int_{D_2(\bar{x})} x_1 \, dx_1 dx_2.$$

In the next examples we will use the number k, which possesses the properties:

(4.5) $$2k^3 - 3k^2 - 3k + 1 = 0, \quad 0 < k < 1.$$

Let $g(k) = 2k^3 - 3k^2 - 3k + 1$. We have: $g(0) > 0$, $g(1) < 0$, $g'(k) = 6k^2 - 6k - 3 < 6k - 6k - 3 < 0$ for all $k \in (0,1)$. So, there exists a unique solution of the equation (4.5), which belongs to the interval $(0,1)$. We denote this solution by the same symbol k.

Example 4.2. *Let $D \subset \mathbb{R}^2_{++}$ be the triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$, that is*

$$D = \left\{ x \in \mathbb{R}^2_{++} : \frac{x_1}{a} + \frac{x_2}{b} \le 1 \right\}.$$

If $\bar{x} \in D$ then we get

$$D_1(\bar{x}) = \left\{ x \in \mathbb{R}^2_{++} : \bar{x}_2 \le x_2 \le \frac{ab\bar{x}_2}{a\bar{x}_2 + b\bar{x}_1}, \ \frac{\bar{x}_1}{\bar{x}_2} x_2 \le x_1 \le a - \frac{a}{b} x_2 \right\},$$

$$D_2(\bar{x}) = \left\{ x \in \mathbb{R}^2_{++} : \bar{x}_1 \le x_1 \le \frac{ab\bar{x}_1}{a\bar{x}_2 + b\bar{x}_1}, \ \frac{\bar{x}_2}{\bar{x}_1} x_1 \le x_2 \le b - \frac{b}{a} x_1 \right\}.$$

Therefore

$$\int_{D_1(\bar{x})} \left\langle \frac{1}{\bar{x}}, x \right\rangle dx = \frac{1}{\bar{x}_2} \int_{\bar{x}_2}^{(ab\bar{x}_2)/(a\bar{x}_2+b\bar{x}_1)} dx_2 \int_{(\bar{x}_1/\bar{x}_2)x_2}^{a-(a/b)x_2} x_2\, dx_1.$$

This reduces to

$$\int_{D_1(\bar{x})} \left\langle \frac{1}{\bar{x}}, x \right\rangle dx = \frac{ab}{6} \frac{\bar{x}_2/b}{(\bar{x}_1/a + \bar{x}_2/b)^2} - \frac{ab}{2} \cdot \frac{\bar{x}_2}{b} + \frac{ab}{3} \cdot \frac{\bar{x}_2}{b} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right).$$

By analogy,

$$\int_{D_2(\bar{x})} \left\langle \frac{1}{\bar{x}}, x \right\rangle dx = \frac{ab}{6} \cdot \frac{\bar{x}_1/a}{(\bar{x}_1/a + \bar{x}_2/b)^2} - \frac{ab}{2} \cdot \frac{\bar{x}_1}{a} + \frac{ab}{3} \cdot \frac{\bar{x}_1}{a} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right).$$

Thus, the sum of these quantities is

$$(4.6)\quad \int_D \varphi \left(\frac{1}{\bar{x}}, x \right) dx = \frac{ab}{6} \cdot \frac{1}{(\bar{x}_1/a + \bar{x}_2/b)} - \frac{ab}{2} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + \frac{ab}{3} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2.$$

Since $A(D) = (ab)/2$ *then for* $\bar{x} \in D$

$$\bar{x} \in Q(D) \iff \frac{1}{3} \frac{1}{(\bar{x}_1/a + \bar{x}_2/b)} - \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + \frac{2}{3} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2 = 1$$

$$\iff 2 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^3 - 3 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2 - 3 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + 1 = 0.$$

Using inequalities $0 < (\bar{x}_1/a + \bar{x}_2/b) \leq 1$ *for* $\bar{x} \in D$ *we get*

$$Q(D) = \left\{ \bar{x} \in \mathbb{R}^2_{++} : \frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} = k \right\},$$

where k *is the solution of (4.5).*

In the more general case we have inequality (see (3.4) and (4.6))

$$f(\bar{x}_1, \bar{x}_2) \leq \frac{6u}{ab(1 - 3u^2 + 2u^3)} \int_D f(x)dx,$$

where $u = u(\bar{x}_1, \bar{x}_2) = \bar{x}_1/a + \bar{x}_2/b < 1$*, function* f *is increasing radiant and integrable on* D*.*

Consider now inequality (3.12) for our triangle D*. We show that*

$$\inf \left\{ \left\langle x, \frac{1}{\bar{x}} \right\rangle^+ : \bar{x} \geq x,\ \bar{x} \in D \right\} = \left(\frac{x_1}{a} + \frac{x_2}{b} \right).$$

Let $\bar{x} = (\bar{x}_1, \bar{x}_2) = (x_1/(x_1/a + x_2/b), x_2/(x_1/a + x_2/b))$*. Then* $\bar{x} \geq x$ *and* $\bar{x} \in D$ *since* $\bar{x}_1/a + \bar{x}_2/b = 1$*. Hence*

$$\inf \left\{ \left\langle x, \frac{1}{\bar{x}} \right\rangle^+ : \bar{x} \geq x,\ \bar{x} \in D \right\} \leq \max \left\{ x_1 \frac{\left(\frac{x_1}{a} + \frac{x_2}{b}\right)}{x_1}, x_2 \frac{\left(\frac{x_1}{a} + \frac{x_2}{b}\right)}{x_2} \right\} = \frac{x_1}{a} + \frac{x_2}{b}.$$

Suppose that the converse inequality does not hold, then $\langle x, 1/\bar{x}\rangle^+ < x_1/a + x_2/b$ *for some* $\bar{x} \geq x$, $\bar{x} \in D$, *hence* $x/(x_1/a + x_2/b) < \bar{x}$. *But this implies that* $\bar{x} \notin D$.

Thus, it follows from (3.12) that

$$\int_D f(x)dx \leq \sup_{y\in D} f(y) \int_D \left(\frac{x_1}{a} + \frac{x_2}{b}\right) dx.$$

Calculation gives the quantity

$$\int_D \left(\frac{x_1}{a} + \frac{x_2}{b}\right) dx = \frac{ab}{3}.$$

Since $A(D) = ab/2$ *then the final result is*

$$\frac{1}{A(D)} \int_D f(x)dx \leq \frac{2}{3} \sup_{y\in D} f(y).$$

Example 4.3. *Now let* Ω *be the triangle from Example 4.2:*

$$\Omega = \left\{x \in \mathbb{R}^2_{++} : \frac{x_1}{a} + \frac{x_2}{b} \leq 1\right\}.$$

Denote by D *the subset of* Ω *such that*

$$\Omega\backslash D = \left\{x \in \Omega : \frac{k}{3} < \frac{x_1}{a}, \frac{k}{3} < \frac{x_2}{b}, \frac{x_1}{a} + \frac{x_2}{b} < k\right\}.$$

Then $(\Omega\backslash D) \subset N(Q(\Omega)) = \{x \in \mathbb{R}^2_{++} : x_1/a + x_2/b \leq k\}$. *Note that* $A(\Omega\backslash D) = (1/18)k^2ab$, *hence* $A(D) = (ab)/2 - (1/18)k^2ab = ab(1/2 -$

$k^2/18)$. *It follows from Proposition 3.4 and formula (4.6) (with* Ω *instead of* D*) that a point* $\bar{x} \in \Omega$ *belongs to* $Q(D)$ *if and only if*

$$\frac{1}{ab(1/2 - k^2/18)} \left[\frac{ab}{6} \frac{1}{(\bar{x}_1/a + \bar{x}_2/b)} - \frac{ab}{2} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + \frac{ab}{3} \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2 \right] = 1$$

$$\iff 2 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^3 - 3 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2 - \left(3 - \frac{k^2}{3} \right) \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + 1 = 0.$$

It is easy to check that there exists a unique solution s *of the equation:*

$$2s^3 - 3s^2 - (3 - k^2/3)s + 1 = 0, \quad 0 < s \leq 1.$$

Hence

$$Q(D) = \left\{ \bar{x} \in \mathbb{R}^2_{++} : \frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} = s \right\}.$$

We may establish also that $s > k$*.*

Remark 4.1. *For any other closed domain* D' *such that* $(\Omega \backslash D') \subset N(Q(\Omega)) = \{x \in \mathbb{R}^2_{++} : x_1/a + x_2/b \leq k\}$ *the set* $Q(D')$ *has the same form, i.e. it is intersection of* $\mathbb{R}^2_{++}$ *and a line* $(\bar{x}_1/a + \bar{x}_2/b) = s'$ *with some* s'*:* $k < s' < 1$*.*

Example 4.4. *Let* Ω *be the same triangle:* $\Omega = \{x \in \mathbb{R}^2_{++} : (x_1/a + x_2/b) \leq 1\}$*. Let* $D \subset \Omega$ *and*

$$\Omega \backslash D = \left\{ x \in \Omega : x_1 < \frac{a}{2},\ x_2 < \frac{b}{2} \right\}.$$

Then $\Omega \backslash D$ *is the normal set, hence* $N(\Omega \backslash D) \cap D = (\Omega \backslash D) \cap D$ *is the empty set. Since* $A(\Omega \backslash D) = ab/4$ *then* $A(D) = ab/2 - ab/4 = ab/4$*. By Proposition*

3.6, we have for $\bar{x} \in D$

$$\bar{x} \in Q(D) \Longleftrightarrow \frac{1}{ab/4}\left[\frac{ab}{6}\frac{1}{(\bar{x}_1/a+\bar{x}_2/b)} - \frac{ab}{2}\left(\frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b}\right)+\frac{ab}{3}\left(\frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b}\right)^2\right]=1$$
$$\Longleftrightarrow 2\left(\frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b}\right)^3 - 3\left(\frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b}\right)^2 - \frac{3}{2}\left(\frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b}\right)+1=0.$$

So,

$$\begin{aligned} Q(D) &= D \cap \left\{\bar{x} \in \mathbb{R}^2_{++} : \frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b} = p\right\} \\ &= \left\{\bar{x} \in \mathbb{R}^2_{++} : \bar{x}_1 \geq \frac{a}{2}, \ \frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b} = p\right\} \\ &\qquad \cup \left\{\bar{x} \in \mathbb{R}^2_{++} : \bar{x}_2 \geq \frac{b}{2}, \ \frac{\bar{x}_1}{a}+\frac{\bar{x}_2}{b} = p\right\}, \end{aligned}$$

where $2p^3 - 3p^2 - (3/2)p + 1 = 0$, $0 < p \leq 1$.

The following two examples were considered in [1] for ICAR functions defined on $\mathbb{R}^2_+$. Note that the coefficient k plays here the same role as the number $(1/3)$ in [1].

Example 4.5. *Consider the triangle D with vertices $(0,0)$, $(a,0)$ and (a,va):*

$$D = \{x \in \mathbb{R}^2_{++} : x_1 \leq a, \ \ x_2 \leq vx_1\}.$$

If $\bar{x} \in D$ then

$$D_1(\bar{x}) = \left\{x \in \mathbb{R}^2_{++} : \bar{x}_1 \leq x_1 \leq a, \ \ \bar{x}_2 \leq x_2 \leq \frac{\bar{x}_2}{\bar{x}_1}x_1\right\},$$

$$D_2(\bar{x}) = \left\{ x \in \mathbb{R}^2_{++} : \bar{x}_1 \le x_1 \le a,\ \frac{\bar{x}_2}{\bar{x}_1} x_1 \le x_2 \le v x_1 \right\}.$$

Calculation gives the following quantities

$$\frac{1}{\bar{x}_2} \int_{D_1(\bar{x})} x_2 dx_1 dx_2 = \frac{1}{\bar{x}_2} \int_{\bar{x}_1}^{a} dx_1 \int_{\bar{x}_2}^{(\bar{x}_2/\bar{x}_1)x_1} x_2 dx_2$$
$$= \bar{x}_2 \left(\frac{a^3}{6\bar{x}_1^2} - \frac{a}{2} + \frac{\bar{x}_1}{3} \right),$$

$$\frac{1}{\bar{x}_1} \int_{D_2(\bar{x})} x_1 \, dx_1 dx_2 = \frac{1}{\bar{x}_1} \int_{\bar{x}_1}^{a} dx_1 \int_{(\bar{x}_2/\bar{x}_1)x_1}^{v x_1} x_1 \, dx_2$$
$$= \left(\frac{va^3}{3\bar{x}_1} - \frac{v\bar{x}_1^2}{3} \right) - \bar{x}_2 \left(\frac{a^3}{3\bar{x}_1^2} - \frac{\bar{x}_1}{3} \right).$$

Further,

$$\int_D \varphi\left(\frac{1}{\bar{x}}, x \right) dx = \left(\frac{va^3}{3\bar{x}_1} - \frac{v\bar{x}_1^2}{3} \right) + \bar{x}_2 \left(\frac{2\bar{x}_1}{3} - \frac{a}{2} - \frac{a^3}{6\bar{x}_1^2} \right).$$

Since $A(D) = va^2/2$ *then a point* $\bar{x} \in D$ *belongs to* $Q(D)$ *if and only if*

$$\left(\frac{2}{3} \frac{a}{\bar{x}_1} - \frac{2}{3} \frac{\bar{x}_1^2}{a^2} \right) + \frac{\bar{x}_2}{va} \left(\frac{4}{3} \frac{\bar{x}_1}{a} - 1 - \frac{1}{3} \frac{a^2}{\bar{x}_1^2} \right) = 1$$
$$\iff \bar{x}_2 \left(1 + 3\frac{\bar{x}_1^2}{a^2} - 4\frac{\bar{x}_1^3}{a^3} \right) = v\bar{x}_1 \left(2 - 3\frac{\bar{x}_1}{a} - 2\frac{\bar{x}_1^3}{a^3} \right).$$

In particular, if $\bar{x}_2 = v\bar{x}_1$ *then we get the equation* $2(\bar{x}_1/a)^3 - 3(\bar{x}_1/a)^2 - 3(\bar{x}_1/a) + 1 = 0$*, hence* $(\bar{x}_1/a) = k$*. So, the point* (ka, vka) *belongs to* $Q(D)$*. This implies that for each* InR *function* f*, which is integrable on* D*:*

$$f(ka, vka) \leq \frac{1}{A(D)} \int_D f(x)dx.$$

If $\bar{x}_2 = v\bar{x}_1/2$ *then then equation has the form* $(\bar{x}_1/a)^2 + 2(\bar{x}_1/a) - 1 = 0$*. This shows that* $(\bar{x}_1/a) = \sqrt{2} - 1$*, therefore* $\left((\sqrt{2}-1)a, v(\sqrt{2}-1)a/2\right) \in Q(D)$*.*

Further, we may set in (3.11) $\bar{x} = (a, va)$*:*

$$\begin{aligned}\int_D f(x)dx &\leq f(a, va) \int_D \max\left\{\frac{x_1}{a}, \frac{x_2}{va}\right\} dx_1 dx_2 \\ &= f(a, va) \int_D \frac{x_1}{a} dx_1 dx_2 \\ &= \frac{f(a, va)}{a} \int_0^a dx_1 \int_0^{vx_1} x_1 dx_2 \\ &= \frac{va^2}{3} f(a, va).\end{aligned}$$

Thus,

$$\frac{1}{A(D)} \int_D f(x)dx \leq \frac{2}{3} f(a, va).$$

Example 4.6. *Let* D *be the square:*

$$D = \{x \in \mathbb{R}^2_{++} : x_1 \leq 1,\ x_2 \leq 1\}.$$

We consider two possible cases for $\bar{x} \in D : (\bar{x}_2/\bar{x}_1) \leq 1$ *and* $(\bar{x}_2/\bar{x}_1) \geq 1$*.*

a) If $(\bar{x}_2/\bar{x}_1) \leq 1$ *then we have*

$$\frac{1}{\bar{x}_2}\int_{D_1(\bar{x})} x_2 dx_1 dx_2 = \frac{1}{\bar{x}_2}\int_{\bar{x}_1}^{1} dx_1 \int_{\bar{x}_2}^{(\bar{x}_2/\bar{x}_1)x_1} x_2\, dx_2$$
$$= \frac{\bar{x}_2}{2}\left(\frac{1}{3\bar{x}_1^2} - 1 + \frac{2\bar{x}_1}{3}\right),$$

$$\frac{1}{\bar{x}_1}\int_{D_2(\bar{x})} x_1 dx_1 dx_2 = \frac{1}{\bar{x}_1}\int_{\bar{x}_1}^{1} dx_1 \int_{(\bar{x}_2/\bar{x}_1)x_1}^{1} x_1\, dx_2$$
$$= \frac{1}{2}\left(\frac{1}{\bar{x}_1} - \bar{x}_1\right) + \frac{\bar{x}_2}{3}\left(\bar{x}_1 - \frac{1}{\bar{x}_1^2}\right).$$

Hence

$$\int_D \varphi\left(\frac{1}{\bar{x}}, x\right) dx = \frac{1}{2}\left(\frac{1}{\bar{x}_1} - \bar{x}_1\right) + \frac{\bar{x}_2}{6}\left(4\bar{x}_1 - 3 - \frac{1}{\bar{x}_1^2}\right).$$

Since $A(D) = 1$ *then we get the equation for* $\bar{x} \in Q(D)$

$$\frac{1}{2}\left(\frac{1}{\bar{x}_1} - \bar{x}_1\right) + \frac{\bar{x}_2}{6}\left(4\bar{x}_1 - 3 - \frac{1}{\bar{x}_1^2}\right) = 1$$
$$\Longleftrightarrow \bar{x}_2\left(1 + 3\bar{x}_1^2 - 4\bar{x}_1^3\right) = 3\bar{x}_1\left(1 - 2\bar{x}_1 - \bar{x}_1^2\right).$$

b) If $(\bar{x}_2/\bar{x}_1) \geq 1$ *then we get the symmetric equation*

$$\bar{x}_1\left(1 + 3\bar{x}_2^2 - 4\bar{x}_2^3\right) = 3\bar{x}_2\left(1 - 2\bar{x}_2 - \bar{x}_2^2\right).$$

Thus, the set $Q(D)$ *can be represented as the union of two sets:*

$$\left\{\bar{x} \in \mathbb{R}^2_{++} : \bar{x}_2 \le \bar{x}_1 \le 1,\ \ \bar{x}_2\left(1 + 3\bar{x}_1^2 - 4\bar{x}_1^3\right) = 3\bar{x}_1\left(1 - 2\bar{x}_1 - \bar{x}_1^2\right)\right\}$$

and

$$\left\{\bar{x} \in \mathbb{R}^2_{++} : \bar{x}_1 \le \bar{x}_2 \le 1,\ \ \bar{x}_1\left(1 + 3\bar{x}_2^2 - 4\bar{x}_2^3\right) = 3\bar{x}_2\left(1 - 2\bar{x}_2 - \bar{x}_2^2\right)\right\}.$$

In particular, if $\bar{x}_1 = \bar{x}_2$ *then*

$$\begin{aligned}\bar{x} \in Q(D) &\iff \left(0 < \bar{x}_1 \le 1,\ \ \left(1 + 3\bar{x}_1^2 - 4\bar{x}_1^3\right) = 3\left(1 - 2\bar{x}_1 - \bar{x}_1^2\right)\right)\\ &\iff \left(0 < \bar{x}_1 \le 1,\ \ 2\bar{x}_1^3 - 3\bar{x}_1^2 - 3\bar{x}_1 + 1 = 0\right).\end{aligned}$$

This implies that $(k, k) \in Q(D)$.

At last we investigate inequality (3.11) with $\bar{x} = (1, 1)$ *for the square* D*:*

$$\int_D f(x)dx \le f(1,1)\int_D \max\{x_1, x_2\}dx_1dx_2.$$

Since $A(D) = 1$ *and*

$$\begin{aligned}\int_D \max\{x_1, x_2\}dx_1dx_2 &= \int_0^1 dx_1 \int_0^{x_1} x_1 dx_2 + \int_0^1 dx_1 \int_{x_1}^1 x_2 dx_2\\ &= \frac{1}{3} + \int_0^1 \frac{\left(1 - x_1^2\right)}{2}dx_1\\ &= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}\end{aligned}$$

then

$$\frac{1}{A(D)}\int_D f(x)dx \leq \frac{2}{3}f(1,1),$$

and this estimate holds for every increasing radiant and integrable on D function f.

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