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HILBERT–PACHPATTE TYPE MULTIDIMENSIONAL INTEGRAL INEQUALITIES

G. D. HANDLEY, J. J. KOLIHA, AND J. PEČARIĆ

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MELBOURNE
MELBOURNE, VIC 3010
AUSTRALIA.
g.handley@pgrad.unimelb.edu.au

j.koliha@ms.unimelb.edu.au

FACULTY OF TEXTILE ENGINEERING
UNIVERSITY OF ZAGREB
10 000 ZAGREB
CROATIA.
pecaric@juda.element.hr

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ABSTRACT. In this paper we use a new approach to obtain a class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte and the present authors.

Key words and phrases: Hilbert's inequality, Hilbert-Pachpatte integral inequalities, Hölder's inequality.

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1. INTRODUCTION

The integral version of Hilbert's inequality [7, Theorem 316] has been generalized in several directions (see [1, 3, 4, 7, 8, 9, 20, 21, 22]). Recently, inequalities similar to those of Hilbert were considered by Pachpatte in [12, 13, 14, 15, 16, 19]. The present authors in [5, 6] established a new class of related inequalities, which were further extended by Dragomir and Kim [2]. Two and higher dimensional variants were treated by Pachpatte in [17, 18]. In the present paper we use a new systematic approach to these inequalities based on Theorem 3.1, which serves as an abstract springboard to classes of concrete inequalities.

To motivate our investigation, we give a typical result of [17]. In this theorem, $H(I \times J)$ denotes the class of functions $u \in C^{(n-1,m-1)}(I \times J)$ such that $D_1^i u(0,t) = 0$, $0 \le i \le n-1$, $t \in J$, $D_2^j u(s,0) = 0$, $0 \le j \le m-1$, $s \in I$, and $D_1^n D_2^{m-1} u(s,t)$ and $D_1^{n-1} D_2^m u(s,t)$ are

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absolutely continuous on $I \times J$. Here $I,\ J$ are intervals of the type $I_\xi = [0,\xi)$ for some real $\xi > 0$.

Theorem 1.1 (Pachpatte [17, Theorem 1])**.** *Let* $u(s,t) \in H(I_x \times I_y)$ *and* $v(k,r) \in H(I_z \times I_w)$. *Then, for* $0 \le i \le n-1$, $0 \le j \le m-1$, *the following inequality holds:*

$$\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{|D_1^i D_2^j u(s,t)\, D_1^i D_2^j v(k,r)|}{s^{2n-2i-1}t^{2m-2j-1} + k^{2n-2i-1}r^{2m-2j-1}}\, dk\, dr \right) ds\, dt$$
$$\le \frac{1}{2}[A_{i,j}B_{i,j}]^2 \sqrt{xyzw} \left(\int_0^x \int_0^y (x-s)(y-t)|D_1^n D_2^m u(s,t)|^2\, ds\, dt \right)^{\frac{1}{2}}$$
$$\times \left(\int_0^z \int_0^w (z-k)(w-r)|D_1^n D_2^m v(k,r)|^2\, dk\, dr \right)^{\frac{1}{2}},$$

where

$$A_{ij} = \frac{1}{(n-i-1)!(m-j-1)!}, \quad B_{ij} = \frac{1}{(2n-2i-1)(2m-2j-1)}.$$

The purpose of the present paper is to obtain a simultaneous generalization of Pachpatte's multivariable results [17], and of the results [5, 6] of the present authors. The single variable results [14, 15, 16, 19] follow as special cases of our theorems. Our treatment is based on Theorem 3.1, in particular on the abstract inequality (3.1), which yields a variety of special cases when the functions Φ_i are specified.

2. Notation and Preliminaries

By $\mathbb{Z}$ ($\mathbb{Z}_+$) and $\mathbb{R}$ ($\mathbb{R}_+$) we denote the sets of all (nonnegative) integers and (nonnegative) real numbers. We will be working with functions of d variables, where d is a fixed positive integer, writing the variable as a vector $s = (s^1, \dots, s^d) \in \mathbb{R}^d$. A multiindex m is an element $m = (m^1, \dots, m^d)$ of $\mathbb{Z}_+^d$. As usual, the factorial of a multiindex m is defined by $m! = m^1! \cdots m^d!$. An integer j may be regarded as the multiindex $(j, \dots, j)$ depending on the context. For vectors in $\mathbb{R}^d$ and multiindices we use the usual operations of vector addition and multiplication of vectors by scalars. We write $s \le \tau$ ($s < \tau$) if $s^j \le \tau^j$ ($s^j < \tau^j$) for $1 \le j \le d$. The same convention will apply to multiindices. In particular, $s \ge 0$ ($s > 0$) will mean $s^j \ge 0$ ($s^j > 0$) for $1 \le j \le d$.

If $s = (s^1, \dots, s^d) \in \mathbb{R}^d$ and $s > 0$, we define the *cell*

$$Q(s) = [0, s^1] \times \cdots \times [0, s^j] \times \cdots \times [0, s^d];$$

replacing the factor $[0, s^j]$ by $\{0\}$ in this product, we get the *face* $\partial_j Q(s)$ of $Q(s)$.

Let $s = (s^1, \dots, s^d)$, $\tau = (\tau^1, \dots, \tau^d) \in \mathbb{R}^d$, $s,\ \tau > 0$, let $k = (k^1, \dots, k^d)$ be a multiindex and let and $u : Q(s) \to \mathbb{R}$. Write $D_j = \frac{\partial}{\partial s^j}$. We use the following notation:

$$s^\tau = (s^1)^{\tau^1} \cdots (s^d)^{\tau^d},$$
$$D^k u(s) = D_1^{k^1} \cdots D_d^{k^d} u(s),$$
$$\int_0^s u(\tau)\, d\tau = \int_0^{s^1} \cdots \int_0^{s^d} u(\tau)\, d\tau^1 \cdots d\tau^d.$$

An exponent $\alpha \in \mathbb{R}$ in the expression s^α, where $s \in \mathbb{R}^d$, will be regarded as a multiexponent, that is, $s^\alpha = s^{(\alpha, \dots, \alpha)}$.

Another positive integer n will be fixed throughout.

The following notation and hypotheses will be used throughout the paper:

$I = \{1, \dots, n\}$	$n \in \mathbb{N}$
$m_i,\ i \in I$	$m_i = (m_i^1, \dots m_i^d) \in \mathbb{Z}_+^d$
$x_i,\ i \in I$	$x_i = (x_i^1, \dots, x_i^d) \in \mathbb{R}^d$, $x_i > 0$
$p_i,\ q_i,\ i \in I$	$p_i,\ q_i \in \mathbb{R}_+$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$
$p,\ q$	$\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}$, $\frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i}$
$a_i,\ b_i,\ i \in I$	$a_i,\ b_i \in \mathbb{R}_+$, $a_i + b_i = 1$
$w_i,\ i \in I$	$w_i \in \mathbb{R}$, $w_i > 0$, $\sum_{i=1}^n w_i = 1$.

Throughout the paper, $u_i,\ v_i,\ \Phi$ will denote functions from $[0, x_i]$ to $\mathbb{R}$ of sufficient smoothness. If m is a multiindex and $x \in \mathbb{R}^d$, $x > 0$, then $C^m[0, x]$ will denote the set of all functions $u : [0, x] \to \mathbb{R}$ which possess continuous derivatives $D^k u$, where $0 \le k \le m$.

The coefficients $p_i,\ q_i$ are conjugate Hölder exponents used in applications of Hölder's inequality, and the coefficients $a_i,\ b_i$ are used in exponents to factorize integrands. The coefficients w_i act as weights in applications of the geometric-arithmetic mean inequality; this enables us to pass from products to sums of terms.

3. THE MAIN RESULT

First we present a theorem that can be regarded as a template for concrete inequalities obtained by selecting suitable functions Φ_i in (3.1). A special case of this theorem is given in [6, Theorem 3.1].

Theorem 3.1. *Let $v_i,\ \Phi_i \in C(Q(x_i))$ and let c_i be multiindices for $i \in I$. If*

$$|v_i(s_i)| \le \int_0^{s_i} (s_i - \tau_i)^{c_i} \Phi_i(\tau_i)\, d\tau_i,\ s_i \in Q(x_i),\ i \in I, \tag{3.1}$$

then

$$\int_0^{x_1} \dots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}}\, ds_1 \cdots ds_n \le U \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i}\, ds_i \right)^{\frac{1}{p_i}}, \tag{3.2}$$

where $\alpha_i = (a_i + b_i q_i) c_i$, $\beta_i = a_i c_i$, and

$$U = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]}.$$

Remark 3.2. Remembering our conventions, we observe that, for example,

$$x_i^{1/q_i} = (x_i^1)^{1/q_i} \dots (x_i^d)^{1/q_i}, \quad \prod_{i=1}^n (\alpha_i + 1)^{1/q_i} = \prod_{i=1}^n \prod_{j=1}^d (\alpha_i^j + 1)^{1/q_i}.$$

Proof. Factorize the integrand on the right side of (3.1) as

$$(s_i - \tau_i)^{(a_i/q_i + b_i) c_i} \cdot (s_i - \tau_i)^{(a_i/p_i) c_i} \Phi_i(\tau_i)$$

and apply Hölder's inequality [10, p. 106] and Fubini's theorem. Then

$$\begin{aligned}|v_i(s_i)| &\le \left(\int_0^{s_i}(s_i-\tau_i)^{(a_i+b_iq_i)c_i}\,d\tau_i\right)^{\frac{1}{q_i}}\\ &\qquad\times\left(\int_0^{s_i}(s_i-\tau_i)^{a_ic_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}}\\ &= \frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i+1)^{1/q_i}}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

Using the inequality of means [10, p. 15]

$$\prod_{i=1}^n s_i^{(\alpha_i+1)/q_i}\le\sum_{i=1}^n w_is_i^{(\alpha_i+1)/(q_iw_i)},$$

we get

$$\prod_{i=1}^n|v_i(s_i)|\le W\sum_{i=1}^n w_is_i^{(\alpha_i+1)/(q_iw_i)}\prod_{i=1}^n\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}},$$

where

$$W=\frac{1}{\prod_{i=1}^n(\alpha_i+1)^{1/q_i}}.$$

In the following estimate we apply Hölder's inequality, Fubini's theorem, and, at the end, change the order of integration:

$$\begin{aligned}&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|v_i(s_i)|}{\sum_{i=1}^n w_is_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n\\ &\qquad\le W\prod_{i=1}^n\left[\int_0^{x_i}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}}\,ds_i\right]\\ &\qquad\le W\prod_{i=1}^n x_i^{1/q_i}\left(\int_0^{x_i}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)\,ds_i\right)^{\frac{1}{p_i}}\\ &\qquad=\frac{W}{\prod_{i=1}^n(\beta_i+1)^{1/p_i}}\prod_{i=1}^n x_i^{1/q_i}\prod_{i=1}^n\left(\int_0^{x_i}(x_i-\tau_i)^{\beta_i+1}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

This proves the theorem. □

If $d=1$ and v_i are replaced by the derivatives $u_i^{(k)}$, the preceding theorem reduces to [6, Theorem 3.1].

Corollary 3.3. *Under the assumptions of Theorem 3.1,*

$$\begin{aligned}\int_0^{x_1}\cdots\int_0^{x_n}&\frac{\prod_{i=1}^n|v_i(s_i)|}{\sum_{i=1}^n w_is_i^{(\alpha_i+1)/(q_iw_i)}}\,ds_1\cdots ds_n\\ &\le p^{1/p}U\prod_{i=1}^n x_i^{1/q_i}\left(\sum_{i=1}^n\frac{1}{p_i}\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\Phi(\tau_i)^{p_i}\,ds_i\right)^{\frac{1}{p}},\end{aligned}\tag{3.3}$$

where U is given by (3.2).

Proof. By the inequality of means, for any $A_i \geq 0$,

$$\prod_{i=1}^{n} A_i^{1/p_i} \leq p^{1/p} \left(\sum_{i=1}^{n} \frac{1}{p_i} A_i \right)^{\frac{1}{p}}.$$

The corollary then follows from the preceding theorem. □

The preceding corollary reduces to [6, Corollary 3.2] in the special case when $d = 1$ and v_i are replaced by $u_i^{(k)}$.

4. APPLICATIONS TO DERIVATIVES

Convention 1. In this section we shall assume that $m_i,\ k_i$ are multiindices satisfying $0 \leq k_i \leq m_i - 1$, and write

$$\alpha_i = (a_i + b_i q_i)(m_i - k_i - 1), \quad \beta_i = a_i(m_i - k_i - 1). \tag{4.1}$$

Recall that according to our conventions, $m_i - k_i - 1 = (m_i^1 - k_i^1 - 1, \ldots, m_1^d - k_i^d - 1)$.

Theorem 4.1. *Let $u_i \in C^{m_i}(Q(x_i))$ be such that $D_j^r u_i(s_i) = 0$ for $s_i \in \partial_j Q(x_i)$, $0 \leq r \leq m_i^j - 1$, $1 \leq j \leq d$, $i \in I$. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |D^{k_i} u_i(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n \leq U_1 \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}, \tag{4.2}$$

where

$$U_1 = \frac{1}{\prod_{i=1}^{n} [(m_i - k_i - 1)!(\alpha_i + 1)^{1/q_i}(\beta_i + 1)^{1/p_i}]}. \tag{4.3}$$

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [11],

$$D^{k_i} u_i(s) = \frac{1}{(m_i - k_i - 1)!} \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} D^{m_i} u_i(\tau_i) \, d\tau_i, \quad i \in I.$$

Inequality (4.2) is proved when we set $v_i(s_i) = D^{k_i} u_i(s_i)$, $c_i = m_i - k_i - 1$, and

$$\Phi_i(s_i) = \frac{|D^{m_i} u_i(s_i)|}{(m_i - k_i - 1)!} \tag{4.4}$$

in Theorem 3.1. □

Corollary 4.2. *Under the hypotheses of Theorem 4.1,*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |D^{k_i} u_i(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n \leq p^{1/p} U_1 \prod_{i=1}^{n} x_i^{1/q_i} \left(\sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p}}, \tag{4.5}$$

where U_1 is given by (4.3).

Proof. The result follows by applying the inequality of means to the preceding theorem. □

Single variable analogues of the preceding two results were obtained in [6, Theorem 4.1] and [6, Corollary 4.2].

We discuss a number of special cases of Theorem 4.1 with similar examples applying also to Corollary 4.2.

Example 4.1. If $a_i = 0$ and $b_i = 1$ for $i \in I$, then (4.2) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_i m_i - q_i k_i - q_i + 1)/(q_i w_i)}}\, ds_1 \cdots ds_n \leq \overline{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}, \tag{4.6}$$

where

$$\overline{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(q_i m_i - q_i k_i - q_i + 1)^{1/q_i}]}. \tag{4.7}$$

Example 4.2. If $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = \frac{1}{n}$, $p_i = \frac{n}{n-1}$, $m_i = m$ and $k_i = k$ for $i \in I$, then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k} u_i(s_i)|}{\sum_{i=1}^n s_i^{nm - nk - n + 1}}\, ds_1 \cdots ds_n \leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (n(m-k-1)+1)} \times \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left| D^{m} u_i(s_i) \right|^{\frac{n}{n-1}} ds_i \right)^{\frac{n-1}{n}}. \tag{4.8}$$

For $d = 2$ and $q = p = n = 2$ this is Pachpatte's theorem [17, Theorem 1] cited in the Introduction; if $d = 1$ and $q = p = n = 2$, we obtain [14, Theorem 1].

Example 4.3. Let $a_i = 1$ and $b_i = 0$ for $i \in I$. Then (4.2) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i - k_i)/(q_i w_i)}}\, ds_1 \cdots ds_n \leq \widetilde{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{m_i - k_i} \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}, \tag{4.9}$$

where

$$\widetilde{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(m_i - k_i)]}. \tag{4.10}$$

Example 4.4. Set $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = \frac{1}{n}$, $p_i = \frac{n}{n-1}$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (4.2) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k} u_i(s_i)|}{\sum_{i=1}^n s_i^{m-k}}\, ds_1 \cdots ds_n \leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (m-k)^n} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{m-k} \left| D^{m} u_i(s_i) \right|^{n/(n-1)} ds_i \right)^{(n-1)/n}. \tag{4.11}$$

In the following theorem we establish another inequality similar to the integral analogue of Hilbert's inequality.

Theorem 4.3. *Let* $u_i \in C^{m_i+1}(Q(x_i))$ *be such that* $D^{m_i}u_i(s_i) = 0$ *for* $s_i \in \partial_j Q(s_i)$, $1 \le j \le d$, $i \in I$. *Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}}\, ds_1 \cdots ds_n \le \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left| D^{m_i+1}u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}. \tag{4.12}$$

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [11] for $m_i = (0, \dots, 0)$:

$$D^{m_i}u_i(s_i) = \int_0^{s_i} D^{m_i+1}u_i(\tau_i)\, d\tau_i, \quad i \in I. \tag{4.13}$$

In Theorem 3.1 set $v_i(s_i) = D^{m_i}u_i(s_i)$, $c_i = 0$, $\Phi_i(s_i) = |D^{m_i+1}u_i(s_i)|$, and the result follows. □

In the special case that $d = 2$, $m_i = (0, 0)$, $p = q = n = 2$, and $w_i = \frac{1}{2}$, the preceding theorem reduces to [17, Theorem 2].

When we apply the inequality of means to the preceding theorem, we get the following corollary which generalizes the inequality obtained in [17, Remark 3].

Corollary 4.4. *Under the hypotheses of Theorem 4.3,*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}}\, ds_1 \cdots ds_n \le p^{1/p} \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i) \left| D^{m_i+1}u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p}}. \tag{4.14}$$

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