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HILBERT–PACHPATTE TYPE MULTIDIMENSIONAL INTEGRAL INEQUALITIES

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Abstract

In this paper we use a new approach to obtain a class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte and the present authors.

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Contents

1. Introduction

The integral version of Hilbert's inequality [7, Theorem 316] has been generalized in several directions (see [1, 3, 4, 7, 8, 9, 20, 21, 22]). Recently, inequalities similar to those of Hilbert were considered by Pachpatte in [12, 13, 14, 15, 16, 19]. The present authors in [5, 6] established a new class of related inequalities, which were further extended by Dragomir and Kim [2]. Two and higher dimensional variants were treated by Pachpatte in [17, 18]. In the present paper we use a new systematic approach to these inequalities based on Theorem 3.1, which serves as an abstract springboard to classes of concrete inequalities.

To motivate our investigation, we give a typical result of [17]. In this theorem, $H(I\times J)$ denotes the class of functions $u\in C^{(n-1,m-1)}(I\times J)$ such that $D_1^i u(0,t)=0$, $0\le i\le n-1$, $t\in J$, $D_2^j u(s,0)=0$, $0\le j\le m-1$, $s\in I$, and $D_1^n D_2^{m-1}u(s,t)$ and $D_1^{n-1}D_2^m u(s,t)$ are absolutely continuous on $I\times J$. Here I, J are intervals of the type $I_\xi=[0,\xi)$ for some real $\xi>0$.

Theorem 1.1 (Pachpatte [17, Theorem 1]). *Let* $u(s,t)\in H(I_x\times I_y)$ *and* $v(k,r)\in H(I_z\times I_w)$. *Then, for* $0\le i\le n-1$, $0\le j\le m-1$, *the following inequality holds:*

$$\int_0^x\int_0^y\left(\int_0^z\int_0^w \frac{|D_1^iD_2^ju(s,t)\,D_1^iD_2^jv(k,r)|}{s^{2n-2i-1}t^{2m-2j-1}+k^{2n-2i-1}r^{2m-2j-1}}\,dk\,dr\right)ds\,dt$$

$$\le \frac{1}{2}[A_{i,j}B_{i,j}]^2\sqrt{xyzw}\left(\int_0^x\int_0^y (x-s)(y-t)|D_1^nD_2^mu(s,t)|^2\,ds\,dt\right)^{\frac12}$$

$$\times\left(\int_0^z\int_0^w (z-k)(w-r)|D_1^nD_2^mv(k,r)|^2\,dk\,dr\right)^{\frac12},$$

where

$$A_{ij} = \frac{1}{(n-i-1)!(m-j-1)!}, \quad B_{ij} = \frac{1}{(2n-2i-1)(2m-2j-1)}.$$

The purpose of the present paper is to obtain a simultaneous generalization of Pachpatte's multivariable results [17], and of the results [5, 6] of the present authors. The single variable results [14, 15, 16, 19] follow as special cases of our theorems. Our treatment is based on Theorem 3.1, in particular on the abstract inequality (3.1), which yields a variety of special cases when the functions Φ_i are specified.

2. Notation and Preliminaries

By $\mathbb{Z}$ ($\mathbb{Z}_+$) and $\mathbb{R}$ ($\mathbb{R}_+$) we denote the sets of all (nonnegative) integers and (nonnegative) real numbers. We will be working with functions of d variables, where d is a fixed positive integer, writing the variable as a vector $s = (s^1, \dots, s^d) \in \mathbb{R}^d$. A multiindex m is an element $m = (m^1, \dots, m^d)$ of $\mathbb{Z}_+^d$. As usual, the factorial of a multiindex m is defined by $m! = m^1! \cdots m^d!$. An integer j may be regarded as the multiindex $(j, \dots, j)$ depending on the context. For vectors in $\mathbb{R}^d$ and multiindices we use the usual operations of vector addition and multiplication of vectors by scalars. We write $s \le \tau$ ($s < \tau$) if $s^j \le \tau^j$ ($s^j < \tau^j$) for $1 \le j \le d$. The same convention will apply to multiindices. In particular, $s \ge 0$ ($s > 0$) will mean $s^j \ge 0$ ($s^j > 0$) for $1 \le j \le d$.

If $s = (s^1, \dots, s^d) \in \mathbb{R}^d$ and $s > 0$, we define the *cell*

$$Q(s) = [0, s^1] \times \cdots \times [0, s^j] \times \cdots \times [0, s^d];$$

replacing the factor $[0, s^j]$ by $\{0\}$ in this product, we get the *face* $\partial_j Q(s)$ of $Q(s)$.

Let $s = (s^1, \dots, s^d)$, $\tau = (\tau^1, \dots, \tau^d) \in \mathbb{R}^d$, $s, \tau > 0$, let $k = (k^1, \dots, k^d)$ be a multiindex and let and $u : Q(s) \to \mathbb{R}$. Write $D_j = \frac{\partial}{\partial s^j}$. We use the following notation:

$$s^\tau = (s^1)^{\tau^1} \cdots (s^d)^{\tau^d},$$

$$D^k u(s) = D_1^{k^1} \cdots D_d^{k^d} u(s),$$

$$\int_0^s u(\tau)\, d\tau = \int_0^{s^1} \cdots \int_0^{s^d} u(\tau)\, d\tau^1 \cdots d\tau^d.$$

An exponent $\alpha \in \mathbb{R}$ in the expression s^α, where $s \in \mathbb{R}^d$, will be regarded as a multiexponent, that is, $s^\alpha = s^{(\alpha,\dots,\alpha)}$.

Another positive integer n will be fixed throughout.

The following notation and hypotheses will be used throughout the paper:

$$
\begin{array}{ll}
I = \{1,\dots,n\} & n \in \mathbb{N} \\
m_i,\ i \in I & m_i = (m_i^1,\dots m_i^d) \in \mathbb{Z}_+^d \\
x_i,\ i \in I & x_i = (x_i^1,\dots,x_i^d) \in \mathbb{R}^d,\, x_i > 0 \\
p_i,\ q_i,\ i \in I & p_i,\ q_i \in \mathbb{R}_+,\ \frac{1}{p_i} + \frac{1}{q_i} = 1 \\
p,\ q & \frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i},\ \frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i} \\
a_i,\ b_i,\ i \in I & a_i,\ b_i \in \mathbb{R}_+,\ a_i + b_i = 1 \\
w_i,\ i \in I & w_i \in \mathbb{R},\ w_i > 0,\ \sum_{i=1}^n w_i = 1.
\end{array}
$$

Throughout the paper, $u_i,\ v_i,\ \Phi$ will denote functions from $[0, x_i]$ to $\mathbb{R}$ of sufficient smoothness. If m is a multiindex and $x \in \mathbb{R}^d$, $x > 0$, then $C^m[0, x]$ will denote the set of all functions $u : [0, x] \to \mathbb{R}$ which possess continuous derivatives $D^k u$, where $0 \le k \le m$.

The coefficients $p_i,\ q_i$ are conjugate Hölder exponents used in applications of Hölder's inequality, and the coefficients $a_i,\ b_i$ are used in exponents to factorize integrands. The coefficients w_i act as weights in applications of the geometric-arithmetic mean inequality; this enables us to pass from products to sums of terms.

3. The Main Result

First we present a theorem that can be regarded as a template for concrete inequalities obtained by selecting suitable functions Φ_i in (3.1). A special case of this theorem is given in [6, Theorem 3.1].

Theorem 3.1. *Let v_i, $\Phi_i \in C(Q(x_i))$ and let c_i be multiindices for $i \in I$. If*

$$|v_i(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{c_i} \Phi_i(\tau_i)\, d\tau_i, \;\; s_i \in Q(x_i), \;\; i \in I, \tag{3.1}$$

then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}}\, ds_1 \cdots ds_n \leq U \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i}\, ds_i \right)^{\frac{1}{p_i}}, \tag{3.2}$$

where $\alpha_i = (a_i + b_i q_i)c_i$, $\beta_i = a_i c_i$, and

$$U = \frac{1}{\prod_{i=1}^n [(\alpha_i+1)^{1/q_i}(\beta_i+1)^{1/p_i}]}.$$

Remark 3.1. *Remembering our conventions, we observe that, for example,*

$$x_i^{1/q_i} = (x_i^1)^{1/q_i} \ldots (x_i^d)^{1/q_i}, \quad \prod_{i=1}^n (\alpha_i + 1)^{1/q_i} = \prod_{i=1}^n \prod_{j=1}^d (\alpha_i^j + 1)^{1/q_i}.$$

Proof. Factorize the integrand on the right side of (3.1) as

$$(s_i - \tau_i)^{(a_i/q_i + b_i)c_i} \cdot (s_i - \tau_i)^{(a_i/p_i)c_i} \Phi_i(\tau_i)$$

and apply Hölder's inequality [10, p. 106] and Fubini's theorem. Then

$$\begin{aligned} |v_i(s_i)| &\le \left(\int_0^{s_i} (s_i - \tau_i)^{(a_i + b_i q_i)c_i} \, d\tau_i \right)^{\frac{1}{q_i}} \\ &\quad \times \left(\int_0^{s_i} (s_i - \tau_i)^{a_i c_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{\frac{1}{p_i}} \\ &= \frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i+1)^{1/q_i}} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{\frac{1}{p_i}} . \end{aligned}$$

Using the inequality of means [10, p. 15]

$$\prod_{i=1}^{n} s_i^{(\alpha_i+1)/q_i} \le \sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)},$$

we get

$$\prod_{i=1}^{n} |v_i(s_i)| \le W \sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)} \prod_{i=1}^{n} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{\frac{1}{p_i}},$$

where

$$W = \frac{1}{\prod_{i=1}^{n} (\alpha_i + 1)^{1/q_i}}.$$

In the following estimate we apply Hölder's inequality, Fubini's theorem, and, at the end, change the order of integration:

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n$$

$$\leq W \prod_{i=1}^n \left[\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{\frac{1}{p_i}} ds_i \right]$$

$$\leq W \prod_{i=1}^n x_i^{1/q_i} \left(\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right) ds_i \right)^{\frac{1}{p_i}}$$

$$= \frac{W}{\prod_{i=1}^n (\beta_i + 1)^{1/p_i}} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - \tau_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{\frac{1}{p_i}}.$$

This proves the theorem. □

If $d = 1$ and v_i are replaced by the derivatives $u_i^{(k)}$, the preceding theorem reduces to [6, Theorem 3.1].

Corollary 3.2. *Under the assumptions of Theorem 3.1,*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n \leq p^{1/p} U \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi(\tau_i)^{p_i} \, ds_i \right)^{\frac{1}{p}}, \tag{3.3}$$

where U is given by (3.2).

Proof. By the inequality of means, for any $A_i \geq 0$,

$$\prod_{i=1}^{n} A_i^{1/p_i} \leq p^{1/p} \left(\sum_{i=1}^{n} \frac{1}{p_i} A_i \right)^{\frac{1}{p}}.$$

The corollary then follows from the preceding theorem. □

The preceding corollary reduces to [6, Corollary 3.2] in the special case when $d = 1$ and v_i are replaced by $u_i^{(k)}$.

4. Applications to Derivatives

Convention 1. *In this section we shall assume that* m_i, k_i *are multiindices satisfying* $0 \le k_i \le m_i - 1$, *and write*

$$\alpha_i = (a_i + b_i q_i)(m_i - k_i - 1), \quad \beta_i = a_i(m_i - k_i - 1). \tag{4.1}$$

Recall that according to our conventions, $m_i - k_i - 1 = (m_i^1 - k_i^1 - 1, \dots, m_1^d - k_i^d - 1)$.

Theorem 4.1. *Let* $u_i \in C^{m_i}(Q(x_i))$ *be such that* $D_j^r u_i(s_i) = 0$ *for* $s_i \in \partial_j Q(x_i)$, $0 \le r \le m_i^j - 1$, $1 \le j \le d$, $i \in I$. *Then*

$$\int_0^{x_1} \dots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n \le U_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}, \tag{4.2}$$

where

$$U_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]}. \tag{4.3}$$

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [11],

$$D^{k_i} u_i(s) = \frac{1}{(m_i - k_i - 1)!} \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} D^{m_i} u_i(\tau_i) \, d\tau_i, \quad i \in I.$$

Inequality (4.2) is proved when we set $v_i(s_i) = D^{k_i}u_i(s_i)$, $c_i = m_i - k_i - 1$, and

$$\Phi_i(s_i) = \frac{|D^{m_i}u_i(s_i)|}{(m_i - k_i - 1)!} \tag{4.4}$$

in Theorem 3.1. □

Corollary 4.2. *Under the hypotheses of Theorem 4.1,*

$$\begin{aligned} \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n \\ \le p^{1/p} U_1 \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left| D^{m_i}u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p}}, \end{aligned} \tag{4.5}$$

where U_1 is given by (4.3).

Proof. The result follows by applying the inequality of means to the preceding theorem. □

Single variable analogues of the preceding two results were obtained in [6, Theorem 4.1] and [6, Corollary 4.2].

We discuss a number of special cases of Theorem 4.1 with similar examples applying also to Corollary 4.2.

Example 4.1. *If $a_i = 0$ and $b_i = 1$ for $i \in I$, then (4.2) becomes*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_i m_i - q_i k_i - q_i + 1)/(q_i w_i)}} \, ds_1 \cdots ds_n \leq \overline{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}, \tag{4.6}$$

where

$$\overline{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(q_i m_i - q_i k_i - q_i + 1)^{1/q_i}]}. \tag{4.7}$$

Example 4.2. *If $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = \frac{1}{n}$, $p_i = \frac{n}{n-1}$, $m_i = m$ and $k_i = k$ for $i \in I$, then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k} u_i(s_i)|}{\sum_{i=1}^n s_i^{nm-nk-n+1}} \, ds_1 \cdots ds_n \leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (n(m-k-1)+1)} \times \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left| D^{m} u_i(s_i) \right|^{\frac{n}{n-1}} ds_i \right)^{\frac{n-1}{n}}. \tag{4.8}$$

For $d = 2$ and $q = p = n = 2$ this is Pachpatte's theorem [17, Theorem 1] cited in the Introduction; if $d = 1$ and $q = p = n = 2$, we obtain [14, Theorem 1].

Example 4.3. *Let $a_i = 1$ and $b_i = 0$ for $i \in I$. Then (4.2) becomes*

$$\text{(4.9)} \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i-k_i)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq \widetilde{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{m_i-k_i} \left| D^{m_i} u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}},$$

where

$$\text{(4.10)} \quad \widetilde{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(m_i - k_i)]}.$$

Example 4.4. *Set $a_i = 0$, $b_i = 1$, $q_i = n$, $w_i = \frac{1}{n}$, $p_i = \frac{n}{n-1}$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (4.2) becomes*

$$\text{(4.11)} \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k} u_i(s_i)|}{\sum_{i=1}^n s_i^{m-k}} ds_1 \cdots ds_n \\ \leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (m-k)^n} \\ \times \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{m-k} \left| D^{m} u_i(s_i) \right|^{n/(n-1)} ds_i \right)^{(n-1)/n}.$$

In the following theorem we establish another inequality similar to the integral analogue of Hilbert's inequality.

Theorem 4.3. *Let $u_i \in C^{m_i+1}(Q(x_i))$ be such that $D^{m_i}u_i(s_i) = 0$ for $s_i \in \partial_j Q(s_i)$, $1 \le j \le d$, $i \in I$. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} \, ds_1 \cdots ds_n \le \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left| D^{m_i+1}u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p_i}}. \tag{4.12}$$

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [11] for $m_i = (0, \dots, 0)$:

$$D^{m_i}u_i(s_i) = \int_0^{s_i} D^{m_i+1}u_i(\tau_i)\, d\tau_i, \quad i \in I. \tag{4.13}$$

In Theorem 3.1 set $v_i(s_i) = D^{m_i}u_i(s_i)$, $c_i = 0$, $\Phi_i(s_i) = |D^{m_i+1}u_i(s_i)|$, and the result follows. □

In the special case that $d = 2$, $m_i = (0,0)$, $p = q = n = 2$, and $w_i = \frac{1}{2}$, the preceding theorem reduces to [17, Theorem 2].

When we apply the inequality of means to the preceding theorem, we get the following corollary which generalizes the inequality obtained in [17, Remark 3].

Corollary 4.4. *Under the hypotheses of Theorem 4.3,*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i}u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} \, ds_1 \cdots ds_n \le p^{1/p} \prod_{i=1}^n x_i^{1/q_i} \left(\sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i) \left| D^{m_i+1}u_i(s_i) \right|^{p_i} ds_i \right)^{\frac{1}{p}}. \tag{4.14}$$

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