

# Journal of Inequalities in Pure and Applied Mathematics

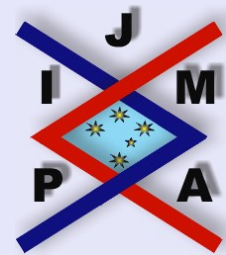
## A GENERALIZATION OF ANDERSSON'S INEQUALITY

A.McD. MERCER

Department of Mathematics and Statistics  
University of Guelph  
Guelph, Ontario K8N 2W1  
Canada.

*EMail:* [amercer@reach.net](mailto:amercer@reach.net)

©2000 Victoria University  
ISSN (electronic): 1443-5756  
129-05



---

volume 6, issue 2, article 57,  
2005.

*Received 21 April, 2005;  
accepted 02 May, 2005.*

*Communicated by: P.S. Bullen*

---

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

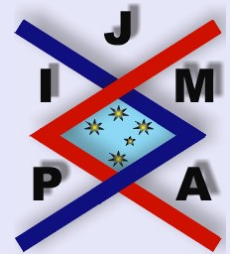
Andersson's Inequality is generalized by replacing the integration there with a positive linear functional which operates on a composition of two functions. These two functions have rather light restrictions and this leads to considerable generalizations of Andersson's result.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Andersson's inequality, Convex functions, Linear functional.

## Contents

1	Introduction .....	3
2	Definitions and Statement of Results .....	5
3	Proofs .....	8
	References	



---

### A Generalization of Andersson's Inequality

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 12

# 1. Introduction

In all that follows we shall use the terms *increasing*, *decreasing*, *positive* and *negative* in the wide sense, meaning *non-decreasing*, *non-increasing*, etc. Andersson [1] or [2, p. 256] showed that if the functions  $f_k$  are convex and increasing in  $[0, 1]$  with  $f_k(0) = 0$  then

$$(1.1) \quad \int_0^1 f_1(x)f_2(x) \cdots f_n(x)dx \geq \frac{2^n}{n+1} \int_0^1 f_1(x)dx \int_0^1 f_2(x)dx \cdots \int_0^1 f_n(x)dx.$$

Then in [3] Fink showed that these hypotheses can be lightened to

$$(1.2) \quad f_k(0) = 0, f_k \in C[0, 1] \text{ and } x^{-1}f_k(x) \text{ is increasing}$$

**Note 1.** In (1.2)  $x^{-1}f_k(x)$  is initially undefined at the origin but since its limit from the right at  $x = 0$  exists, this can be taken as its definition there.

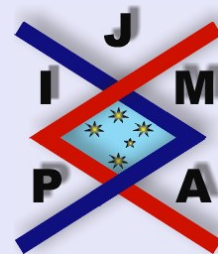
**Note 2.** That the hypotheses in (1.2) are lighter than those used by Andersson can be seen immediately from the convexity condition

$$f(x) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \quad \text{if } 0 < a \leq x \leq b$$

by letting  $a \rightarrow 0$ .

An interesting special case of (1.1) is obtained by taking all the  $f_k$  to be the same function  $f$ , when we get

$$(1.3) \quad \int_0^1 f^n(x)dx \geq \frac{2^n}{n+1} \left( \int_0^1 f(x)dx \right)^n, \quad n = 1, 2, \dots$$



---

A Generalization of  
Andersson's Inequality

A.McD. Mercer

---

Title Page

Contents



Go Back

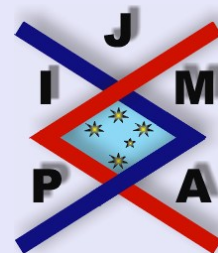
Close

Quit

Page 3 of 12

and the obvious question arises here concerning the case of non-integral  $n$ .

It is the purpose of this paper to generalize Andersson's result in a way that involves positive linear functionals. With this aim in mind, in the next section we make some preparations and then state our theorems.



---

## A Generalization of Andersson's Inequality

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 4 of 12

## 2. Definitions and Statement of Results

Let the functions  $f_k$  satisfy (1.2) and  $L$  denote a positive linear functional defined on  $C[0, 1]$ . We introduce a second set of functions  $\phi_k$  defined by

$$\phi_k = e_1 \frac{L(f_k)}{L(e_1)}, \quad \text{where } e_1(x) = x.$$

Finally we let  $F_k$  denote functions defined and differentiable on the ranges of  $f_k$  and  $\phi_k$ . (If only one of each of the functions above is involved in certain places we shall omit the subscript). We now introduce our two theorems.

Theorem 2.1 will be a generalization of the special case (1.3) and Theorem 2.2 will be a generalization of (1.1). We choose to proceed in this order since, on the one hand, Theorem 2.1 is of interest in its own right and, secondly, once it is proved, it is a simple matter to prove Theorem 2.2.

**Theorem 2.1.** *With  $f$  satisfying (1.2) and  $\phi$ ,  $F$  and  $L$  being as introduced above we have:*

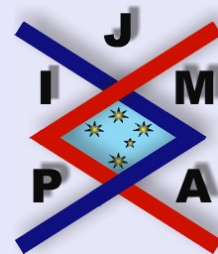
(a) *If  $F'$  and  $g$  are increasing then*

$$(2.1) \quad L[F(f)g] \geq L[F(\phi)g].$$

(b) *If  $F'$  and  $g$  are decreasing then*

$$L[F(f)g] \leq L[F(\phi)g].$$

**Theorem 2.2.** *With  $f_k$  satisfying (1.2) and  $\phi_k$ ,  $F_k$  and  $L$  being as introduced above we have*



---

A Generalization of  
Andersson's Inequality

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 5 of 12

(a) If all the  $F_k$  and  $F'_k$  are increasing then

$$L \left[ \prod_{k=1}^n F_k(f_k) \right] \geq L \left[ \prod_{k=1}^n F_k(\phi_k) \right]$$

and

(b) If all the  $F_k$  and  $F'_k$  are decreasing then

$$L \left[ \prod_{k=1}^n F_k(f_k) \right] \leq L \left[ \prod_{k=1}^n F_k(\phi_k) \right]$$

Before proceeding we give an example of Theorem 2.1.

**Example 2.1.** In Theorem 2.1 take  $F(u) = u^\alpha$ ,  $g(u) = 1$  and let  $L$  be defined by

$$L(w) = \int_0^1 w(t) dt.$$

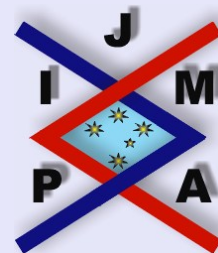
Then

(a)

$$(2.2) \quad \int_0^1 f^\alpha(x) dx \geq \frac{2^\alpha}{\alpha + 1} \left( \int_0^1 f(x) dx \right)^\alpha$$

for  $-1 < \alpha \leq 0$  or  $\alpha \geq 1$

and



A Generalization of  
Andersson's Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

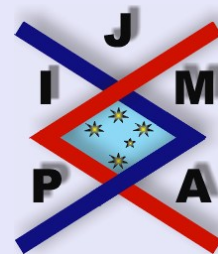
Page 6 of 12

(b)

$$(2.3) \quad \int_0^1 f^\alpha(x) dx \leq \frac{2^\alpha}{\alpha+1} \left( \int_0^1 f(x) dx \right)^\alpha \quad \text{for } 0 \leq \alpha \leq 1.$$

The values of  $\alpha$  are determined by the behaviour of  $F'$  (except that the condition  $-1 < \alpha$  is required to ensure the convergence of the integral on the left).

The above example answers the question which arose at (1.3).



---

**A Generalization of  
Andersson's Inequality**

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 7 of 12

### 3. Proofs

First we need two lemmas.

**Lemma 3.1.** *Let  $p, q \in C[0, 1]$ , and  $L$  be a positive linear functional. Suppose that  $L(p) = 0$  and that  $p(x)$  changes sign once, from negative to positive, in the interval and suppose that  $q(x)$  is increasing there. Then*

(a)

$$L(pq) \geq 0$$

(b) *If  $q(x)$  is decreasing then the inequality is to be reversed.*

*Proof of Lemma 3.1(a).* If  $q(x)$  is constant the result is trivial. Otherwise there is  $\gamma \in (0, 1)$  such that  $p(\gamma) = 0$ .

Then, defining

$$p_1(x) = \min(0, p(x)) : p_2(x) = \max(0, p(x)) \quad \text{in } [0, 1]$$

we have

$$p_1(x)q(x) \geq p_1(x)q(\gamma) \quad \text{in } [0, \gamma]$$

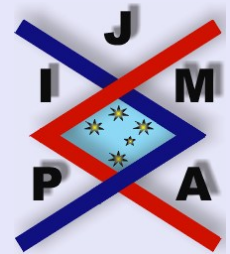
and

$$p_2(x)q(x) \geq p_2(x)q(\gamma) \quad \text{in } [\gamma, 1]$$

So

$$\begin{aligned} L(pq) &= L(p_1q) + L(p_2q) \\ &\geq L(p_1q(\gamma)) + L(p_2q(\gamma)) \\ &= q(\gamma)L(p) = 0 \end{aligned}$$

which completes the proof of (a). The proof of Lemma 3.1(b) is similar.  $\square$



---

A Generalization of  
Andersson's Inequality

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 8 of 12



The next lemma was proved in [3] for the case in which  $L$  is integration over  $[0, 1]$ . Here we give a different proof which refers to a general positive linear functional.

**Lemma 3.2.** *With  $f$  and  $\phi$  as above we have*

(a)

$$L[(f - \phi)g] \equiv L \left[ \left( f - e_1 \frac{L(f)}{L(e_1)} \right) g \right] \geq 0 \quad \text{for all } f$$

*if  $g$  is increasing. (If  $g$  is constant or if  $f = \phi$  we will have equality)*

(b) *The inequality is to be reversed if  $g$  is decreasing.*

*Proof of Lemma 3.2(a).* First we observe that the difference

$$f(x) - \phi(x) \equiv f(x) - x \frac{L(f)}{L(e_1)}$$

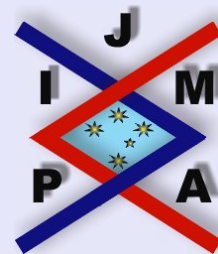
changes sign in  $(0, 1)$  because  $x^{-1}f(x)$  is increasing and both

$$f(x) - x \frac{L(f)}{L(e_1)} > 0 \quad \text{and} \quad f(x) - x \frac{L(f)}{L(e_1)} < 0 \quad \text{in } 0 \leq x \leq 1$$

are seen to be impossible, on operating through with  $L$ . Clearly, this sign change is from minus to plus.

It is also clear that

$$L(f - \phi) = 0$$



A Generalization of  
Andersson's Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 9 of 12

and so the result follows on taking

$$f - \phi = p \text{ and } g = q$$

in Lemma 3.1(a). The proof of part (b) is similar. □

*Proof of Theorem 2.1(a).*  $F'$  and  $g$  are increasing functions. Then

$$(3.1) \quad [F(f(x)) - F(\phi(x))]g(x) = [f(x) - \phi(x)]Q(x)g(x),$$

where

$$Q(x) \equiv \frac{1}{[f(x) - \phi(x)]} \int_{\phi(x)}^{f(x)} F'(t) dt$$

It is a simple matter to see that this quotient is increasing with  $x$ .

In fact, it is obvious since  $Q(x)$  is the average value of the increasing function  $F'$  over the interval  $(f(x), \phi(x))$  [or  $(\phi(x), f(x))$ ], each of whose end-points moves to the right with increasing  $x$ .

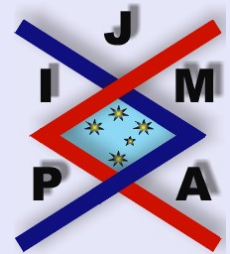
Since  $Qg$  is an increasing function, then from (3.1) we get

$$L[F(f)g - F(\phi)g] = L[(f - \phi)Qg] \geq 0$$

on applying Lemma 3.2 to the right hand side.

This concludes the proof of Theorem 2.1(a) and the proof of part (b) is similar. □

**Note 3.** The function  $g$  played no significant part in this proof but its presence is needed when we come to deduce Theorem 2.2 from Theorem 2.1.




---

**A Generalization of  
Andersson's Inequality**

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 10 of 12

*Proof of Theorem 2.2(a).* For the sake of brevity we shall take  $n = 3$  because this will indicate the method of proof for any  $n > 1$ .

We have

$$L[F_1(f_1)F_2(f_2)F_3(f_3)] \geq L[F_1(f_1)F_2(f_2)F_3(\phi_3)]$$

on reading  $F_1(f_1)F_2(f_2)$  as  $g$  in (2.1).

Then

$$L[F_1(f_1)F_2(f_2)F_3(\phi_3)] \geq L[F_1(f_1)F_2(\phi_2)F_3(\phi_3)]$$

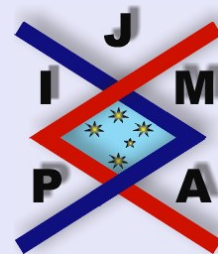
on reading  $F_1(f_1)F_3(\phi_3)$  as  $g$  in (2.1).

Finally

$$L[F_1(f_1)F_2(\phi_2)F_3(\phi_3)] \geq L[F_1(\phi_1)F_2(\phi_2)F_3(\phi_3)]$$

on reading  $F_2(\phi_2)F_3(\phi_3)$  as  $g$  in (2.1).

The general case is proved in exactly the same way. This concludes the proof of Theorem 2.2(a) and that of part (b) is similar.  $\square$



A Generalization of  
Andersson's Inequality

A.McD. Mercer

Title Page

Contents



Go Back

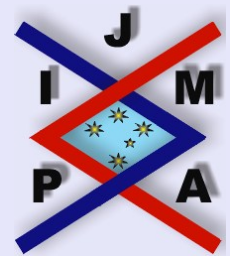
Close

Quit

Page 11 of 12

## References

- [1] B.X. ANDERSSON, An inequality for convex functions, *Nordisk Mat. Tidsk*, **6** (1958), 25–26.
- [2] D.S. MITRINOVIĆ, J.E PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer, 1993.
- [3] A.M. FINK, Andersson's Inequality, *Math. Inequal. & Applics*, **6**(3) (2003), 241–245.



---

### A Generalization of Andersson's Inequality

A.McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 12 of 12