Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 4, Article 89, 2004

# NEW INEQUALITIES ON POLYNOMIAL DIVISORS

LAURENŢIU PANAITOPOL AND DORU ŞTEFĂNESCU

UNIVERSITY OF BUCHAREST
010014 BUCHAREST 1
ROMANIA.
pan@al.math.unibuc.ro

UNIVERSITY OF BUCHAREST
P.O. BOX 39–D5
BUCHAREST 39
ROMANIA.
stef@fpcm5.fizica.unibuc.ro

*Received 06 May, 2004; accepted 27 June, 2004*
*Communicated by L. Toth*

ABSTRACT. In this paper there are obtained new bounds for divisors of integer polynomials, deduced from an inequality on Bombieri's $l_2$–weighted norm [1]. These bounds are given by explicit limits for the size of coefficients of a divisor of given degree. In particular such bounds are very useful for algorithms of factorization of integer polynomials.

*Key words and phrases:* Inequalities, Polynomials.

2000 *Mathematics Subject Classification.* 12D05, 12D10, 12E05, 26C05.

## 1. INTRODUCTION

Let $P$ be a nonconstant polynomial in $\mathbb{Z}[X]$ and suppose that $Q$ is a nontrivial divisor of $P$ over $\mathbb{Z}$. In many problems it is important to have a priori information on $Q$. For example in polynomial factorization a key step is the determination of an upper bound for the coefficients of such a polynomial $Q$ in function of the coefficients and the degree finding (see J. von zur Gathen [3], M. van Hoeij [4]). Throughout this paper we will consider inequalities involving the quadratic norm, Bombieri's norm and the height of a polynomial.

We derive upper bounds for the coefficients of a divisor in function of the weighted $l_2$–norm of E. Bombieri. Our main result is Theorem 3.1 in which we obtain upper bounds for the size of polynomial coefficients of prescribed degree of a given polynomial over the integers. This may lead to a significant reduction of the factorization cost. In particular we obtain bounds for the heights which are an improvement on an inequality of B. Beauzamy [2].

We first present some definitions.

ISSN (electronic): 1443-5756

131-04

**Definition 1.1.** Let $P(X) = \sum_{j=0}^{n} a_j X^j \in \mathbb{C}[X]$. The quadratic norm of $P$ is

$$||P|| \;=\; \sqrt{\sum_{j=0}^{n} |a_j|^2}.$$

The weighted $l_2$–norm of Bombieri is

$$[P]_2 \;=\; \sqrt{\sum_{j=0}^{n} \frac{|a_j|^2}{\binom{n}{j}}}.$$

The height of $P$ is

$$\mathrm{H}(P) = \max\bigl\{|a_0|, |a_1|, \dots, |a_n|\bigr\}.$$

The measure of $P$ is

$$\mathrm{M}(P) = \exp\left\{\int_0^1 \log\bigl|P\left(e^{2i\pi t}\right)\bigr| dt\right\}.$$

Note that

$$\mathrm{H}(P) \le \binom{n}{\lfloor n/2 \rfloor} \cdot \mathrm{M}(P), \quad ||P|| \le \binom{2n}{n}^{\frac{1}{2}} \cdot \mathrm{M}(P), \quad \mathrm{H}(P) \le 2^n \cdot \mathrm{M}(P).$$

Bombieri's norm and the height are used in estimations of the absolute values of the coefficients of polynomial divisors of integer polynomials. This reduces to the evaluation of the height of the divisors. We mention the evaluation of B. Beauzamy:

- If $P(X) = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X]$, $n \ge 1$ and $Q$ is a divisor of $P$ in $\mathbb{Z}[X]$, then

$$\mathrm{H}(Q) \;\le\; \frac{3^{3/4} \cdot 3^{n/2}}{2(\pi\, n)^{1/2}} [P]_2. \tag{1.1}$$

  (B. Beauzamy [2]).

## 2. INEQUALITIES ON FACTORS OF COMPLEX POLYNOMIALS

We derive inqualities on the coefficients of divisors of complex polynomials, using a well–known inequality on Bombieri's norm [1] and an idea of B. Beauzamy [2].

**Proposition 2.1.** *If*

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{C}[X] \setminus \mathbb{C},$$

$P(0) \neq 0$, $n \ge 3$ *and*

$$Q(X) = b_d X^d + b_{n-1} X^{d-1} + \cdots + b_1 X + b_0 \in \mathbb{C}[X]$$

*is a nontrivial divisor of* $P$ *of degree* $d \ge 2$, *then*

$$\left(\frac{|a_0|^2}{|b_0|^2} + \frac{|a_n|^2}{|b_d|^2}\right)\left(|b_0|^2 + |b_d|^2 + \frac{|b_i|^2}{\binom{d}{i}}\right) \le \binom{n}{d}[P]_2^2, \quad \textit{for all} \quad i = 1, 2, \dots, d-1.$$

*Proof.* By an inequality of B. Beauzamy, E. Bombieri, P. Enflo and H. Montgomery [1] (cf. also B. Beauzamy [2]), it is known that if $P = QR$ in $\mathbb{C}[X]$, then

$$\binom{n}{d}^{\frac{1}{2}}[P]_2 \geq [Q]_2[R]_2. \tag{2.1}$$

Note that

$$[R]_2^2 \geq |R(0)|^2 + |lc(R)|^2 = \frac{|a_0|^2}{|b_0|^2} + \frac{|a_n|^2}{|b_d|^2}.$$

Therefore, by (2.1),

$$[P]_2 \geq \frac{\left(\frac{|a_0|^2}{|b_0|^2} + \frac{|a_n|^2}{|b_d|^2}\right)[Q]_2}{\binom{n}{d}^{\frac{1}{2}}}. \tag{2.2}$$

But a lower bound for $[Q]_2$ is $\sqrt{|b_0|^2 + |b_d|^2 + \frac{|b_i|^2}{\binom{d}{i}}}$. Therefore

$$\left(\frac{|a_0|^2}{|b_0|^2} + \frac{|a_n|^2}{|b_d|^2}\right)\left(|b_0|^2 + b|_d|^2 + \frac{|b_i|^2}{\binom{d}{i}}\right) \leq \binom{n}{d}[P]_2^2.$$

$\square$

**Corollary 2.2.** *For all* $i \in \{1, 2, \ldots, d-1\}$ *we have*

$$|b_i| \leq \sqrt{\binom{d}{i}\binom{n}{d}\left(\frac{|a_0|^2}{|b_0|^2} + \frac{|a_n|^2}{|b_d|^2}\right)^{-1}[P]_2^2 - \binom{d}{i}(|b_0|^2 + |b_d|^2)}.$$

## 3. BOUNDS FOR DIVISORS OF INTEGER POLYNOMIALS

For polynomials with integer cofficients Corollary 2.2 allows us to give upper bounds for the heights of polynomial divisors.

**Theorem 3.1.** *Let* $P(X) = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X] \setminus \mathbb{Z}$ *and let* $Q(X) = \sum_{i=0}^{d} a_i X^i \in \mathbb{Z}[X]$ *be a nontrivial divisor of* $P$ *in* $\mathbb{Z}[X]$*, with* $1 \leq d \leq n-1$*. If* $n = \deg(P) \geq 4$ *and* $P(0) \neq 0$*, then*

$$|b_i| \leq \sqrt{\binom{d}{i}\left(\frac{1}{2}\binom{n}{d}[P]_2^2 - a_0^2 - a_n^2\right)} \quad \textit{for all} \quad i. \tag{3.1}$$

*Proof.* We consider first the case $d = 1$. We have $\binom{d}{i} = 1$ and $i = 0$ or $i = 1$. Therefore $b_i$ divides $a_0$ or $a_n$, so

$$b_i^2 \leq a_0^2 + a_n^2 \,.$$

As $n \geq 4$ it follows that

$$b_i^2 \leq 2(a_0^2 + a_n^2) - (a_0^2 + a_n^2) \leq \frac{1}{2}n[P]_2^2 - a_0^2 - a_n^2 \,.$$

Consider now $d \geq 2$.

For $i = 0$ we have

$$\begin{aligned} b_0^2 \le a_0^2 \le a_0^2 + a_n^2 &= \frac{1}{2} 4(a_0^2 + a_n^2) - a_0^2 - a_n^2 \\ &\le \frac{1}{2} n(a_0^2 + a_n^2) - a_0^2 - a_n^2 \\ &\le \frac{1}{2}\binom{n}{d}(a_0^2 + a_n^2) - a_0^2 - a_n^2 \\ &\le \binom{d}{i}\left(\frac{1}{2}\binom{n}{d}(a_0^2 + a_n^2) - a_0^2 - a_n^2\right). \end{aligned}$$

The same argument holds for $i = d$.

We suppose now $1 \le i \le d-1$. First we consider the case

$$\left|\frac{a_0}{b_0}\right| = \left|\frac{a_n}{b_d}\right| = 1\,.$$

We have

$$\left(\frac{a_0}{b_0}\right)^2 + \left(\frac{a_n}{b_d}\right)^2 = 2$$

and the inequality follows from Corollary 2.2.

If

$$\left|\frac{a_0}{b_0}\right| > 1 \quad \text{or} \quad \left|\frac{a_n}{b_d}\right| > 1,$$

we have

$$\left(\frac{a_0}{b_0}\right)^2 + \left(\frac{a_n}{b_d}\right)^2 \ge 5$$

and by Proposition 2.1 we have

$$b_i^2 \le \binom{d}{i}\left(\frac{1}{5}\binom{n}{d}[P]_2^2 - b_0^2 - b_d^2\right).$$

To conclude, it is sufficient to prove that

$$\frac{1}{5}\binom{n}{d}[P]_2^2 - b_0^2 - b_d^2 \le \frac{1}{2}\binom{n}{d}[P]_2^2 - a_0^2 - a_n^2\,,$$

i.e.

$$\left(\frac{1}{2} - \frac{1}{5}\right)\binom{n}{d}[P]_2^2 \ \ge\ a_0^2 + a_n^2 - b_0^2 - b_d^2\,,$$

which follows from

$$\frac{3}{10} n[P]_2^2 \ge \frac{12}{10}\left(a_0^2 + a_n^2\right) > a_0^2 + a_n^2 - b_0^2 - b_d^2.$$

□

**Corollary 3.2.** *If* $n = \deg(P) \ge 4$ *and* $d = \deg(Q)$ *we have*

$$\mathrm{H}(Q) \le \sqrt{\binom{d}{\lfloor d/2 \rfloor}} \cdot \sqrt{\frac{1}{2}\binom{n}{d}[P]_2^2 - a_0^2 - a_n^2}\,.$$

**Corollary 3.3.** *If* $n = \deg(P) \ge 6$ *we have*

$$\mathrm{H}(Q) \le \sqrt{\frac{1}{2}\binom{d}{\lfloor d/2 \rfloor} \cdot \binom{n}{d}[P]_2^2 - 2(a_0^2 + a_n^2)}\,.$$

*Proof.* For $d = \deg(Q) = 1$ we put $Q(X) = b_0 + b_1 X$. Then $b_0$ divides $a_0$ and $b_1$ divides $a_n$. So

$$\mathrm{H}(Q)^2 < a_0^2 + a_n^2 \leq \frac{n-4}{2}(a_0^2 + a_n^2)\,.$$

But this is equivalent to the statement.
For $d \geq 2$ we have $\binom{d}{\lfloor d/2 \rfloor} \geq 2$ and the inequality follows by Corollary 3.2. □

**Corollary 3.4.** *For* $n = \deg(P) \geq 6$ *we have*

$$\mathrm{H}(Q) \;\leq\; \sqrt{\frac{3^{(2n+3)/2}}{4\pi\, n}[P]_2^2 - a_0^2 - a_n^2}\,.$$

*Proof.* By a B. Beauzamy result we have

$$\frac{1}{2}\binom{d}{\lfloor d/2 \rfloor} \;\leq\; \frac{3^{(2n+3)/2}}{4\pi\, n}\,.$$

□

**Corollary 3.5.** *If* $\deg(P) \geq 6$ *we have*

$$\mathrm{H}(Q) \;\leq\; \sqrt{\frac{3^{(2n+3)/2}}{4\pi\, n}[P]_2^2 - 2(a_0^2 + a_n^2)}\,.$$

*Proof.* We use Corollary 3.3 and the proof of Corollary 3.4. □

## 4. EXAMPLES

We compare now the various results throughout the paper. We also compare them with estimates of B. Beauzamy [2]. The computations are done using the gp–package.

### 4.1. Prescribed coefficients.

In polynomial factorization we are ultimately interested in knowing the size of coefficients of an arbitrary divisor of prescribed degree. We consider the folllowing bounds for the $i$th coefficient of a divisor of degree $d$ of the polynomial $P$:

$$B_1(P,d,i) \;=\; \sqrt{\tfrac{1}{2}\tbinom{d}{i}\cdot\tbinom{n}{d}}\,[P]_2 \qquad \text{(B. Beauzamy [2]),}$$

$$B_2(P,d,i) \;=\; \sqrt{\tbinom{d}{i}}\cdot\sqrt{\tfrac{1}{2}\tbinom{n}{d}[P]_2^2 - a_0^2 - a_n^2} \qquad \text{(Theorem 3.1).}$$

Let

$$\begin{aligned}
Q_1 &= x^4 + x + 1,\\
Q_2 &= 7x^5 + 12x^4 + 11,\\
Q_3 &= 11x^7 - x^5 + x + 1,\\
Q_4 &= 111x^7 - x^5 + x^3 + x + 2,\\
Q_5 &= 3x^7 + 12x^6 - x + 37,\\
Q_6 &= 4x^{11} + x^8 + 8x^7 - x^5 + x^3 + x + 2,\\
Q_7 &= 113x^{11} + 2x^9 - 13x^8 + x^7 - x^4 + 3x^2 + 2x + 91,\\
Q_8 &= x^{15} + 30x^4 + 5x^3 + 2x^2 + 5x + 2.
\end{aligned}$$

| $P$ | $d$ | $i$ | $B_1(P,d,i)$ | $B_2(P,d,i)$ |
|---|---|---|---|---|
| $Q_1$ | 3 | 0 | 2.12 | 1.58 |
| $Q_1$ | 3 | 1 | 3.67 | 2.73 |
| $Q_1$ | 3 | 2 | 3.67 | 2.73 |
| $Q_1$ | 3 | 3 | 2.12 | 1.58 |
| $Q_2$ | 3 | 0 | 31.52 | 28.70 |
| $Q_2$ | 3 | 1 | 54.60 | 49.71 |
| $Q_3$ | 5 | 0 | 35.81 | 34.07 |
| $Q_3$ | 5 | 1 | 80.09 | 76.19 |
| $Q_3$ | 5 | 2 | 113.26 | 107.79 |
| $Q_3$ | 6 | 0 | 20.68 | 17.48 |
| $Q_3$ | 6 | 1 | 50.65 | 42.82 |
| $Q_3$ | 6 | 2 | 80.09 | 67.71 |
| $Q_3$ | 6 | 3 | 92.48 | 78.18 |
| $Q_4$ | 6 | 5 | 508.75 | 429.97 |
| $Q_5$ | 6 | 1 | 171.38 | 145.27 |
| $Q_6$ | 9 | 1 | 70.88 | 69.59 |
| $Q_6$ | 9 | 3 | 216.54 | 212.62 |
| $Q_6$ | 10 | 1 | 33.41 | 30.27 |
| $Q_6$ | 10 | 4 | 153.11 | 138.72 |
| $Q_7$ | 8 | 2 | 6973.46 | 6931.07 |
| $Q_7$ | 10 | 2 | 2282.60 | 2064.71 |
| $Q_8$ | 13 | 1 | 71.15 | 70.70 |
| $Q_8$ | 13 | 5 | 708.01 | 703.45 |
| $Q_8$ | 14 | 2 | 71.15 | 67.88 |
| $Q_8$ | 14 | 3 | 142.31 | 135.77 |
| $Q_8$ | 14 | 6 | 408.77 | 389.97 |

Table 1

4.2. **Divisors of prescribed degree.** We consider now bounds for divisors of given degree $d$. Let

$$B_1(P,d) = \sqrt{\frac{1}{2}\binom{d}{\lfloor d/2\rfloor}\binom{n}{d}} \cdot [P]_2 \qquad \text{(B. Beauzamy [2]),}$$

$$B_2(P,d) = \sqrt{\binom{d}{\lfloor d/2\rfloor}} \cdot \sqrt{\frac{1}{2}\binom{n}{d}[P]_2^2 - a_0^2 - a_n^2} \qquad \text{(Corollary 3.2),}$$

$$B_3(P,d) = \sqrt{\frac{1}{2}\binom{d}{\lfloor d/2\rfloor} \cdot \binom{n}{d}[P]_2^2 - 2(a_0^2 + a_n^2)} \qquad \text{(Corollary 3.3)}$$

We have $B_3(P,d) < B_2(P,d) < B_1(P,d)$. The bounds $B_2(P,d)$ and $B_3(P,d)$ are better for polynomials with large leading coefficients and and large free terms.

Considering the polynomials

$$R_1 = x^5 + 13x^4 + x + 101,$$
$$R_2 = 11x^7 - x^5 + x + 1,$$
$$R_3 = 11x^7 - x^5 + x + 34,$$
$$R_4 = 14x^{11} - 3x^2 + x + 29,$$
$$R_5 = 12x^{15} - x^{14} + x^{12} - x^{11} + 2x^9 + 5x^4 + 5x^3 + 2x^2 + 5x + 16,$$

we obtain

| $P$ | $d$ | $B_1(P,d)$ | $B_2(P,d)$ | $B_3(P,d)$ |
|---|---|---|---|---|
| $R_1$ | 1 | 159.96 | 143.96 | — |
| $R_1$ | 2 | 319.93 | 303.57 | — |
| $R_1$ | 3 | 391.84 | 371.80 | — |
| $R_1$ | 4 | 391.84 | 350.61 | — |
| $R_2$ | 4 | 113.26 | 111.64 | 109.99 |
| $R_2$ | 5 | 113.26 | 110.54 | 107.74 |
| $R_3$ | 4 | 366.20 | 360.93 | 355.58 |
| $R_4$ | 2 | 238.84 | 236.66 | 234.46 |
| $R_4$ | 9 | 1895.80 | 1878.49 | 1861.02 |
| $R_4$ | 10 | 1199.01 | 1143.22 | 1084.57 |
| $R_5$ | 1 | 54.89 | 53.04 | 51.12 |
| $R_5$ | 2 | 205.41 | 204.43 | 203.45 |
| $R_5$ | 12 | 9190.89 | 9180.83 | 9170.76 |
| $R_5$ | 13 | 6016.85 | 5988.26 | 5959.54 |
| $R_5$ | 14 | 3216.14 | 3107.60 | 2995.12 |

Table 2

4.3. **Arbitrary divisors.** Finally we consider bounds for an arbitrary divisor of a polynomial $P$. We put

$$B_1(P) = \frac{3^{3/4}\cdot 3^{n/2}}{2(\pi n)^{1/2}}\cdot [P]_2 \qquad \text{(B. Beauzamy [2])},$$

$$B_2(P) = \sqrt{\frac{3^{(2n+3)/2}}{4\pi\, n}[P]_2^2 - a_0^2 - a_n^2} \qquad (n \ge 4,\ \text{Corollary 3.4}),$$

$$B_3(P) = \sqrt{\frac{3^{(2n+3)/2}}{4\pi\, n}[P]_2^2 - 2(a_0^2 + a_n^2)} \qquad (n \ge 6,\ \text{Corollary 3.5}).$$

We always have $B_3(P) < B_2(P) < B_1(P)$.
If we consider

$$R_6 = 12x^6 - 2x^4 + x + 11,$$
$$R_7 = x^6 - x^3 + 11,$$
$$R_8 = 2x^6 - x^3 + 114,$$
$$R_9 = 2x^9 + x^5 + 11,$$
$$R_{10} = 2x^{11} - x^6 + x^5 + 119.$$

we get

| $P$ | $B_1(P)$ | $B_2(P)$ | $B_3(P)$ |
|---|---|---|---|
| $R_6$ | 115.47 | 114.33 | 113.16 |
| $R_7$ | 78.30 | 77.52 | 76.73 |
| $R_8$ | 808.15 | 800.07 | 791.90 |
| $R_9$ | 336.22 | 336.02 | 335.85 |
| $R_{10}$ | 9712.13 | 9711.41 | 9710.68 |

Table 3

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