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ON A Q-ANALOGUE OF SÁNDOR'S FUNCTION

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Abstract

In this paper we obtain a q-analogue of J. Sándor's theorems [6], on employing the q-analogue of Stirling's formula established by D. S. Moak [5].

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Dedicated to Professor Katsumi Shiratani on the occasion of his 71st birthday

Contents

1. Introduction

F. H. Jackson defined a q-analogue of the gamma function which extends the q-factorial

$$(n!)_q = 1(1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{n-1}), \text{ cf. [3, 4]},$$

which becomes the ordinary factorial as $q \to 1$. He defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0<q<1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, \qquad q>1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

It is well-known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary gamma function. In [2], R. Askey obtained a q-analogue of many of the classical facts about the gamma function.

In his interesting paper [6], J. Sándor defined the functions S and S_* by

$$S(x) = \min\{m \in N : x \le m!\}, \qquad x \in (1,\infty),$$

and

$$S_*(x) = \max\{m \in N : m! \le x\}, \qquad x \in [1,\infty).$$

He has studied many important properties of S_* and proved the following theorems:

Theorem 1.1.

$$S_*(x) \sim \frac{\log x}{\log \log x} \qquad (x \to \infty).$$

Theorem 1.2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

In [1], C. Adiga and T. Kim have obtained a generalization of Theorems 1.1 and 1.2.

We now define the q-analogues of S and S_* as follows:

$$S_q(x) = \min\{m \in N : x \leq \Gamma_q(m+1)\}, \qquad x \in (1, \infty),$$

and

$$S_q^*(x) = \max\{m \in N : \Gamma_q(m+1) \leq x\}, \qquad x \in [1, \infty),$$

where $0 < q < 1$.

Clearly $S_q(x) \to S(x)$ and $S_q^*(x) \to S_*(x)$ as $q \to 1^-$.

In Section 2 of this paper we study some properties of S_q and S_q^*, which are similar to those of S and S_* studied by Sándor [6]. In Section 3 we prove two theorems which are the q-analogues of Theorems 1.1 and 1.2 of Sándor [6].

To prove our main theorems we make use of the following q-analogue of Stirling's formula established by D.S. Moak [5]:

$$\log\Gamma_q(z) \sim \left(z-\frac{1}{2}\right)\log\left(\frac{q^z-1}{q-1}\right)+\frac{1}{\log q}\int_{-\log q}^{-z\log^n q}\frac{u\,du}{e^u-1} + C_q+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log q}{q^z-1}\right)^{2k-1} q^z\, P_{2k-1}(q^z), \tag{1.1}$$

where C_q is a constant depending upon q, and $P_n(z)$ is a polynomial of degree n satisfying,

$$P_n(z)=(z-z^2)P'_{n-1}(z)\ +\ (nz+1)P_{n-1}(z), \qquad P_0=1, \quad n\geq 1.$$

2. Some Properties of S_q and S_q^*

From the definitions of S_q and S_q^*, it is clear that

$$S_q(x) = m \qquad \text{if } x \in (\Gamma_q(m), \Gamma_q(m+1)], \qquad \text{for } m \geq 2, \tag{2.1}$$

and

$$S_q^*(x) = m \qquad \text{if } x \in [\Gamma_q(m+1), \Gamma_q(m+2)), \qquad \text{for } m \geq 1. \tag{2.2}$$

(2.1) and (2.2) imply

$$S_q(x) = \begin{cases} S_q^*(x)+1, & \text{if } x \in (\Gamma_q(k+1), \Gamma_q(k+2)), \\ S_q^*(x), & \text{if } x = \Gamma_q(k+2). \end{cases}$$

Thus

$$S_q^*(x) \leq S_q(x) \leq S_q^*(x) + 1.$$

Hence it suffices to study the function S_q^*. The following are the simple properties of S_q^*.

(1) S_q^* is surjective and monotonically increasing.

(2) S_q^* is continuous for all $x \in [1,\infty)\backslash A$, where $A = \{\Gamma_q(k+1) : k \geq 2\}$. Since

$$\lim_{x\to\Gamma_q(k+1)^+} S_q^*(x) = k \quad \text{and} \quad \lim_{x\to\Gamma_q(k+1)^-} S_q^*(x) = (k-1),\ (k \geq 2),$$

S_q^* is continuous from the right at $x = \Gamma_q(k+1)$, $k \geq 2$, but it is not continuous from the left.

(3) S_q^* is differentiable on $(1,\infty)\backslash A$ and since

$$\lim_{x\to\Gamma_q(k+1)^+}\frac{S_q^*(x)-S_q^*(\Gamma_q(k+1))}{x-\Gamma_q(k+1)}=0$$

for $k\geq 1$, it has a right derivative in $A\cup\{1\}$.

(4) S_q^* is Riemann integrable on $[a,b]$, where $\Gamma_q(k+1)\leq a<b, k\geq 1$.

(i) If $[a,b]\subset[\Gamma_q(k+1),\Gamma_q(k+2)], k\geq 1$, then

$$\int_a^b S_q^*(x)dx=\int_a^b kdx=k(b-a).$$

(ii) For $n>k$, we have

$$\begin{aligned}\int_{\Gamma_q(k+1)}^{\Gamma_q(n+1)} S_q^*(x)dx &= \sum_{m=1}^{(n-k)}\int_{\Gamma_q(k+m)}^{\Gamma_q(k+m+1)} S_q^*(x)dx\\ &= \sum_{m=1}^{(n-k)}(k+m-1)[\Gamma_q(k+m+1)-\Gamma_q(k+m)]\\ &= \sum_{m=1}^{(n-k)}(k+m-1)\Gamma_q(k+m)[q+q^2+\cdots+q^{k+m-1}].\end{aligned}$$

(iii) If $a \in [\Gamma_q(k+1), \Gamma_q(k+2))$ and $b \in [\Gamma_q(n), \Gamma_q(n+1))$ then

$$\begin{aligned}
\int_a^b S_q^*(x)dx \\
&= \int_a^{\Gamma_q(k+2)} S_q^*(x)dx + \int_{\Gamma_q(k+2)}^{\Gamma_q(n)} S_q^*(x)dx + \int_{\Gamma_q(n)}^{b} S_q^*(x)dx \\
&= k[\Gamma_q(k+2) - a] + \sum_{m=1}^{n-k-2} (k+m)\Gamma_q(k+m+1) \\
&\qquad \times (q + q^2 + \ldots + q^{k+m}) + (n-1)[b - \Gamma_q(n)],
\end{aligned}$$

by (ii).

3. Main Theorems

We now prove our main theorems.

Theorem 3.1. *If* $0 < q < 1$, *then*

$$S_q^*(x) \sim \frac{\log x}{\log\left(\frac{1}{1-q}\right)}.$$

Proof. If $\Gamma_q(n+1) \le x < \Gamma_q(n+2)$, then

$$\log \Gamma_q(n+1) \le \log x < \log \Gamma_q(n+2). \tag{3.1}$$

By (1.1) we have

$$\log \Gamma_q(n+1) \sim \left(n + \frac{1}{2}\right) \log\left(\frac{q^{n+1}-1}{q-1}\right) \sim n \log\left(\frac{1}{1-q}\right). \tag{3.2}$$

Dividing (3.1) throughout by $n \log\left(\frac{1}{1-q}\right)$, we obtain

$$\frac{\log \Gamma_q(n+1)}{n \log\left(\frac{1}{1-q}\right)} \le \frac{\log x}{S_q^*(x)\ \log\left(\frac{1}{1-q}\right)} < \frac{\log \Gamma_q(n+2)}{n \log\left(\frac{1}{1-q}\right)}. \tag{3.3}$$

Using (3.2) in (3.3) we deduce

$$\lim_{n\to\infty} \frac{\log x}{S_q^*(x) \log\left(\frac{1}{1-q}\right)} = 1.$$

This completes the proof. □

Theorem 3.2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n(S_q^*(n))^\alpha} \tag{3.4}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \le 1$.

Proof. Since

$$S_q^*(x) \sim \frac{\log x}{\log\left(\frac{1}{1-q}\right)},$$

we have

$$A\frac{\log n}{\log\left(\frac{1}{1-q}\right)} < S_q^*(n) < B\frac{\log n}{\log\left(\frac{1}{1-q}\right)},$$

for all $n \ge N > 1$, $A, B > 0$. Therefore to examine the convergence or divergence of the series (3.4) it suffices to study the series

$$\log\left(\frac{1}{1-q}\right)\sum_{n=1}^{\infty} \frac{1}{n(\log n)^\alpha}\ .$$

By the integral test, $\sum \frac{1}{n(\log n)^\alpha}$ converges for $\alpha > 1$ and diverges for $0 \le \alpha \le 1$. If $\alpha < 0$, then $\frac{1}{n(\log n)^\alpha} > \frac{1}{n}$ for $n \ge 3$. Hence $\sum \frac{1}{(n\log n)^\alpha}$ diverges by the comparison test. □

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