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ON THE DEGREE OF STRONG APPROXIMATION OF CONTINUOUS FUNCTIONS BY SPECIAL MATRIX

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Abstract: In the presented paper we will generalize the result of L. Leindler [3] to the class $MRBVS$ and extend it to the strong summability with a mediate function satisfying the standard conditions.

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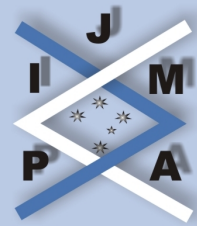
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1. Introduction

Let f be a continuous and 2π -periodic function and let

$$(1.1) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Denote by $S_n(x) = S_n(f, x)$ the n -th partial sum of (1.1) and by $\omega(f, \delta)$ the modulus of continuity of $f \in C_{2\pi}$. The usual supremum norm will be denoted by $\|\cdot\|$.

Let $A := (a_{nk})$ ($k, n = 0, 1, \dots$) be a lower triangular infinite matrix of real numbers satisfying the following conditions:

$$(1.2) \quad a_{nk} \geq 0 \quad (0 \leq k \leq n), \quad a_{nk} = 0, \quad (k > n) \quad \text{and} \quad \sum_{k=0}^n a_{nk} = 1,$$

where $k, n = 0, 1, 2, \dots$

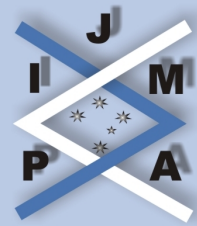
Let the A -transformation of $(S_n(f; x))$ be given by

$$(1.3) \quad t_n(f) := t_n(f; x) := \sum_{k=0}^n a_{nk} S_k(f; x) \quad (n = 0, 1, \dots)$$

and the strong A_r -transformation of $(S_n(f; x))$ for $r > 0$ be given by

$$T_n(f, r) := T_n(f, r; x) := \left\{ \sum_{k=0}^n a_{nk} |S_k(f; x) - f(x)|^r \right\}^{\frac{1}{r}} \quad (n = 0, 1, \dots).$$

Now we define two classes of sequences.



A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$(1.4) \quad \sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c) c_m$$

for $m = 0, 1, 2, \dots$, where $K(c)$ is a constant depending only on c (see [3]).

A null sequence $c := (c_n)$ of positive numbers is called of Mean Rest Bounded Variation, or briefly $c \in MRBVS$, if it has the property

$$(1.5) \quad \sum_{n=2m}^{\infty} |c_n - c_{n+1}| \leq K(c) \frac{1}{m+1} \sum_{n=m}^{2m} c_n$$

for $m = 0, 1, 2, \dots$ (see [5]).

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n , where $K(\alpha_n)$ denotes the sequence of constants appearing in the inequalities (1.4) or (1.5) for the sequence $\alpha_n := (a_{nk})_{k=0}^{\infty}$. Now we can give some conditions to be used later on. We assume that for all n

$$(1.6) \quad \sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{nm} \quad (0 \leq m \leq n)$$

and

$$(1.7) \quad \sum_{k=2m}^{\infty} |a_{nk} - a_{nk+1}| \leq K \frac{1}{m+1} \sum_{k=m}^{2m} a_{nk} \quad (0 \leq 2m \leq n)$$

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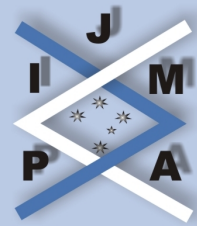


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hold if $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$ or $MRBVS$, respectively.

In [1] and [2] P. Chandra obtained some results on the degree of approximation for the means (1.3) with a mediate function H such that:

$$(1.8) \quad \int_u^{\pi} \frac{\omega(f; t)}{t^2} dt = O(H(u)) \quad (u \rightarrow 0_+), \quad H(t) \geq 0$$

and

$$(1.9) \quad \int_0^t H(u) du = O(tH(t)) \quad (t \rightarrow O_+).$$

In [3], L. Leindler generalized this result to the class $RBVS$. Namely, he proved the following theorem:

Theorem 1.1. *Let (1.2), (1.6), (1.8) and (1.9) hold. Then for $f \in C_{2\pi}$*

$$\|t_n(f) - f\| = O(a_{n0}H(a_{n0})).$$

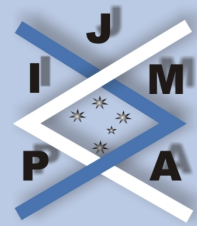
It is clear that

$$(1.10) \quad RBVS \subseteq MRBVS.$$

In [7], we proved that $RBVS \neq MRBVS$. Namely, we showed that the sequence

$$d_n := \begin{cases} 1 & \text{if } n = 1, \\ \frac{1+m+(-1)^n m}{(2^{\mu_m})^2 m} & \text{if } \mu_m \leq n < \mu_{m+1}, \end{cases}$$

where $\mu_m = 2^m$ for $m = 1, 2, 3, \dots$, belongs to the class $MRBVS$ but it does not belong to the class $RBVS$.



In the present paper we will generalize the mentioned result of L. Leindler [3] to the class $MRBVS$ and extend it to strong summability with a mediate function H defined by the following conditions:

$$(1.11) \quad \int_u^\pi \frac{\omega^r(f; t)}{t^2} dt = O(H(r; u)) \quad (u \rightarrow 0_+), \quad H(t) \geq 0 \text{ and } r > 0,$$

and

$$(1.12) \quad \int_0^t H(r; u) du = O(tH(r; t)) \quad (t \rightarrow 0_+).$$

By K_1, K_2, \dots we shall denote either an absolute constant or a constant depending on the indicated parameters, not necessarily the same in each occurrence.

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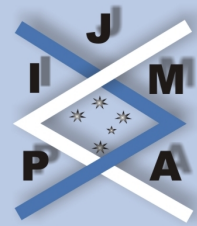
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2. Main Results

Our main results are the following.

Theorem 2.1. *Let (1.2), (1.7) and (1.11) hold. Then for $f \in C_{2\pi}$ and $r > 0$*

$$(2.1) \quad \|T_n(f, r)\| = O\left(\left\{a_{n0}H\left(r; \frac{\pi}{n}\right)\right\}^{\frac{1}{r}}\right).$$

If, in addition (1.12) holds, then

$$(2.2) \quad \|T_n(f, r)\| = O\left(\{a_{n0}H(r; a_{n0})\}^{\frac{1}{r}}\right).$$

Using the inequality

$$\|t_n(f) - f\| \leq \|T_n(f, 1)\|,$$

we can formulate the following corollary.

Corollary 2.2. *Let (1.2), (1.7) and (1.11) hold. Then for $f \in C_{2\pi}$*

$$\|t_n(f) - f\| = O\left(a_{n0}H\left(1; \frac{\pi}{n}\right)\right).$$

If, in addition (1.12) holds, then

$$\|t_n(f) - f\| = O(a_{n0}H(1; a_{n0})).$$

Remark 1. By the embedding relation (1.7) we can observe that Theorem 1.1 follows from Corollary 2.2.

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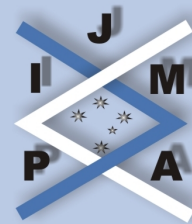
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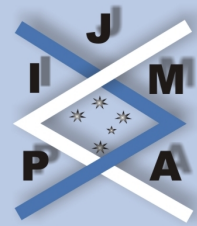
For special cases, putting

$$H(r; t) = \begin{cases} t^{r\alpha-1} & \text{if } \alpha r < 1, \\ \ln \frac{\pi}{t} & \text{if } \alpha r = 1, \\ K_1 & \text{if } \alpha r > 1, \end{cases}$$

where $r > 0$ and $0 < \alpha \leq 1$, we can derive from Theorem 2.1 the next corollary.

Corollary 2.3. *Under the conditions (1.2) and (1.7) we have, for $f \in C_{2\pi}$ and $r > 0$,*

$$\|T_n(f, r)\| = \begin{cases} O(\{a_{n0}\}^\alpha) & \text{if } \alpha r < 1, \\ O\left(\left\{\ln\left(\frac{\pi}{a_{n0}}\right) a_{n0}\right\}^\alpha\right) & \text{if } \alpha r = 1, \\ O\left(\{a_{n0}\}^{\frac{1}{r}}\right) & \text{if } \alpha r > 1. \end{cases}$$



3. Lemmas

To prove our main result we need the following lemmas.

Lemma 3.1 ([6]). *If (1.11) and (1.12) hold, then for $r > 0$*

$$\int_0^s \frac{\omega^r(f; t)}{t} dt = O(sH(r; s)) \quad (s \rightarrow 0_+).$$

Lemma 3.2. *If (1.2) and (1.7) hold, then for $f \in C_{2\pi}$ and $r > 0$*

$$(3.1) \quad \|T_n(f, r)\|_C \leq O\left(\left\{\sum_{k=0}^n a_{nk} E_k^r(f)\right\}^{\frac{1}{r}}\right),$$

where $E_n(f)$ denotes the best approximation of the function f by trigonometric polynomials of order at most n .

Proof. It is clear that (3.1) holds for $n = 0, 1, \dots, 5$. Namely, by the well known inequality [8]

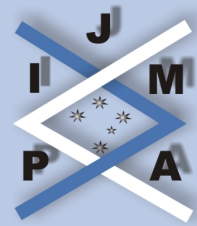
$$(3.2) \quad \|\sigma_{n,m} - f\| \leq 2 \frac{n+1}{m+1} E_n(f) \quad (0 \leq m \leq n),$$

where

$$\sigma_{n,m}(f; x) = \frac{1}{m+1} \sum_{k=n-m}^n S_k(f; x),$$

for $m = 0$, we obtain

$$\{T_n(f, r; x)\}^r \leq 12^r \sum_{k=0}^n a_{nk} E_k^r(f)$$



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and (3.1) is obviously valid, for $n \leq 5$.

Let $n \geq 6$ and let $m = m_n$ be such that

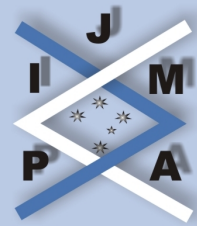
$$2^{m+1} + 4 \leq n < 2^{m+2} + 4.$$

Hence

$$\begin{aligned} \{T_n(f, r; x)\}^r &\leq \sum_{k=0}^3 a_{nk} |S_k(f; x) - f(x)|^r \\ &+ \sum_{k=1}^{m-1} \sum_{i=2^{k+2}}^{2^{k+1}+4} a_{ni} |S_i(f; x) - f(x)|^r + \sum_{k=2^m+5}^n a_{nk} |S_k(f; x) - f(x)|^r. \end{aligned}$$

Applying the Abel transformation and (3.2) to the first sum we obtain

$$\begin{aligned} &\{T_n(f, r; x)\}^r \\ &\leq 8^r \sum_{k=0}^3 a_{nk} E_k^r(f) + \sum_{k=1}^{m-1} \left(\sum_{i=2^{k+2}}^{2^{k+1}+3} (a_{ni} - a_{n,i+1}) \sum_{l=2^k+2}^i |S_l(f; x) - f(x)|^r \right. \\ &\quad \left. + a_{n,2^{k+1}+4} \sum_{i=2^k+2}^{2^{k+1}+4} |S_i(f; x) - f(x)|^r \right) \\ &+ \sum_{k=2^m+2}^{n-1} (a_{nk} - a_{n,k+1}) \sum_{l=2^{m-1}}^k |S_l(f; x) - f(x)|^r \\ &+ a_{nn} \sum_{k=2^m+2}^n |S_k(f; x) - f(x)|^r \end{aligned}$$



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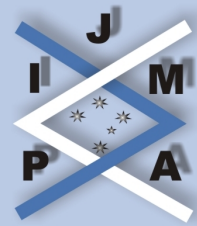
$$\begin{aligned} &\leq 8^r \sum_{k=0}^3 a_{nk} E_k^r(f) + \sum_{k=1}^{m-1} \left(\sum_{i=2^{k+2}}^{2^{k+1}+3} |a_{ni} - a_{n,i+1}| \sum_{l=2^{k+2}}^{2^{k+1}+3} |S_l(f; x) - f(x)|^r \right. \\ &\quad \left. + a_{n,2^{k+1}+4} \sum_{i=2^{k+2}}^{2^{k+1}+4} |S_i(f; x) - f(x)|^r \right) \\ &\quad + \sum_{k=2^m+2}^{n-1} |a_{nk} - a_{n,k+1}| \sum_{l=2^m+2}^{2^{m+2}+3} |S_l(f; x) - f(x)|^r \\ &\quad + a_{nn} \sum_{k=2^m+2}^{2^{m+2}+4} |S_k(f; x) - f(x)|^r. \end{aligned}$$

Using the well-known Leindler's inequality [4]

$$\left\{ \frac{1}{m+1} \sum_{k=n-m}^n |S_k(f; x) - f(x)|^s \right\}^{\frac{1}{s}} \leq K_1 E_{n-m}(f)$$

for $0 \leq m \leq n$, $m = O(n)$ and $s > 0$, we obtain

$$\begin{aligned} \{T_n(f, r; x)\}^r &\leq 8^r \sum_{k=0}^3 a_{nk} E_k^r(f) \\ &\quad + K_2 \left\{ \sum_{k=1}^{m-1} \left((2^k + 3) E_{2^{k+2}}^r(f) \left(\sum_{i=2^{k+2}}^{2^{k+1}+3} |a_{ni} - a_{n,i+1}| + a_{n,2^{k+1}+4} \right) \right) \right. \\ &\quad \left. + 3(2^m + 1) E_{2^m+2}^r \left(\sum_{k=2^m+2}^{n-1} |a_{nk} - a_{n,k+1}| + a_{nn} \right) \right\}. \end{aligned}$$



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Using (1.7) we get

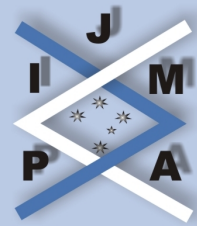
$$\begin{aligned} \{T_n(f, r; x)\}^r &\leq 8^r \sum_{k=0}^3 a_{nk} E_k^r(f) \\ &+ K_2 \left\{ \sum_{k=1}^{m-1} \left((2^k + 3) E_{2^{k+2}}^r(f) \left(K \frac{1}{2^{k-1} + 2} \sum_{i=2^{k-1}+1}^{2^{k+2}} a_{ni} + a_{n,2^{k+1}+4} \right) \right) \right. \\ &\quad \left. 3(2^m + 1) E_{2^{m+2}}^r(f) \left(K \frac{1}{2^{m-1} + 2} \sum_{i=2^{m-1}+1}^{2^{m+2}} a_{ni} + a_{nn} \right) \right\}. \end{aligned}$$

In view of (1.7), we also obtain for $1 \leq k \leq m - 1$,

$$\begin{aligned} a_{n,2^{k+1}+4} &= \sum_{i=2^{k+1}+4}^{\infty} (a_{ni} - a_{ni+1}) \leq \sum_{i=2^{k+1}+4}^{\infty} |a_{ni} - a_{ni+1}| \\ &\leq \sum_{i=2^{k+2}}^{\infty} |a_{ni} - a_{ni+1}| \leq K \frac{1}{2^{k-1} + 2} \sum_{i=2^{k-1}+1}^{2^{k+2}} a_{ni} \end{aligned}$$

and

$$\begin{aligned} a_{nn} &= \sum_{i=n}^{\infty} (a_{ni} - a_{ni+1}) \leq \sum_{i=n}^{\infty} |a_{ni} - a_{ni+1}| \\ &\leq \sum_{i=2^{m+2}}^{\infty} |a_{ni} - a_{ni+1}| \leq K \frac{1}{2^{m-1} + 2} \sum_{i=2^{m-1}+1}^{2^{m+2}} a_{ni}. \end{aligned}$$



Hence

$$\begin{aligned} \{T_n(f, r; x)\}^r &\leq 8^r \sum_{k=0}^3 a_{nk} E_k^r(f) \\ &\quad + K_3 \left\{ \sum_{k=1}^{m-1} E_{2^{k+2}}^r(f) \sum_{i=2^{k-1}+1}^{2^{k+2}} a_{ni} + E_{2^{m+2}}^r(f) \sum_{i=2^{m-1}+1}^{2^{m+2}} a_{ni} \right\} \\ &\leq 8^r \sum_{k=0}^3 a_{nk} E_k^r(f) + 2K_3 \sum_{k=3}^{2^m+2} a_{nk} E_k^r(f) \\ &\leq K_4 \sum_{k=0}^n a_{nk} E_k^r(f). \end{aligned}$$

This ends our proof. □

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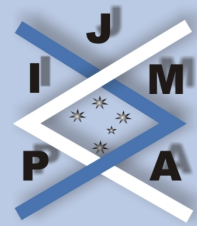
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4. Proof of Theorem 2.1

Using Lemma 3.2 we have

$$(4.1) \quad |T_n(f, r; x)| \leq K_1 \left\{ \sum_{k=0}^n a_{nk} E_k^r(f) \right\}^{\frac{1}{r}} \\ \leq K_2 \left\{ \sum_{k=0}^n a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) \right\}^{\frac{1}{r}}.$$

If (1.7) holds, then, for any $m = 1, 2, \dots, n$,

$$a_{nm} - a_{n0} \leq |a_{nm} - a_{n0}| = |a_{n0} - a_{nm}| = \left| \sum_{k=0}^{m-1} (a_{nk} - a_{nk+1}) \right| \\ \leq \sum_{k=0}^{m-1} |a_{nk} - a_{nk+1}| \leq \sum_{k=0}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{n0},$$

whence

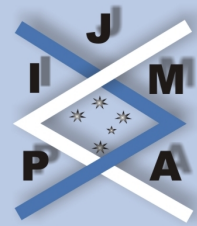
$$(4.2) \quad a_{nm} \leq (K + 1) a_{n0}.$$

Therefore, by (1.2),

$$(4.3) \quad (K + 1)(n + 1) a_{n0} \geq \sum_{k=0}^n a_{nk} = 1.$$

First we prove (2.1). Using (4.2), we get

$$\sum_{k=0}^n a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) \leq (K + 1) a_{n0} \sum_{k=0}^n \omega^r \left(f; \frac{\pi}{k+1} \right)$$



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$$\begin{aligned} &\leq K_3 a_{n0} \int_1^{n+1} \omega^r \left(f; \frac{\pi}{t} \right) dt \\ &= \pi K_3 a_{n0} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega^r (f; u)}{u^2} du \end{aligned}$$

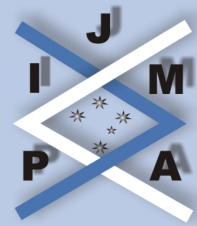
and by (4.1), (1.11) we obtain that (2.1) holds.

Now, we prove (2.2). From (4.3) we obtain

$$\begin{aligned} &\sum_{k=0}^n a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) \\ &\leq \sum_{k=0}^{\left[\frac{1}{(K+1)a_{n0}} \right] - 1} a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) + \sum_{k=\left[\frac{1}{(K+1)a_{n0}} \right] - 1}^n a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right). \end{aligned}$$

Again using (1.2), (4.2) and the monotonicity of the modulus of continuity, we get

$$\begin{aligned} &\sum_{k=0}^n a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) \\ &\leq (K+1) a_{n0} \sum_{k=0}^{\left[\frac{1}{(K+1)a_{n0}} \right] - 1} \omega^r \left(f; \frac{\pi}{k+1} \right) \\ &\quad + K_4 \omega^r (f; \pi (K+1) a_{n0}) \sum_{k=\left[\frac{1}{(K+1)a_{n0}} \right] - 1}^n a_{nk} \end{aligned}$$



$$\begin{aligned} &\leq K_5 a_{n0} \int_1^{\frac{1}{(K+1)a_{n0}}} \omega^r \left(f; \frac{\pi}{t} \right) dt + K_4 \omega^r \left(f; \pi (K+1) a_{n0} \right) \\ (4.4) \quad &\leq K_6 \left(a_{n0} \int_{a_{n0}}^{\pi} \frac{\omega^r (f; u)}{u^2} du + \omega^r (f; a_{n0}) \right). \end{aligned}$$

Moreover

$$\begin{aligned} (4.5) \quad \omega^r (f; a_{n0}) &\leq 4^r \omega^r \left(f; \frac{a_{n0}}{2} \right) \\ &\leq 2 \cdot 4^r \int_{\frac{a_{n0}}{2}}^{a_{n0}} \frac{\omega^r (f; t)}{t} dt \\ &\leq 2 \cdot 4^r \int_0^{a_{n0}} \frac{\omega^r (f; t)}{t} dt. \end{aligned}$$

Thus collecting our partial results (4.1), (4.4), (4.5) and using (1.11) and Lemma 3.1 we can see that (2.2) holds. This completes our proof. \square

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