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# SOME ESTIMATIONS FOR THE INTEGRAL TAYLOR'S REMAINDER

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ABSTRACT. In this paper, using Leibnitz's formula and pre-Grüss inequality we prove some inequalities involving Taylor's remainder.

*Key words and phrases:* Taylor's remainder, Leibnitz's formula, Pre-Grüss inequality.

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## 1. INTRODUCTION

Recently, H. Gauchman ([1] – [2]) derived new types of inequalities involving Taylor's remainder.

In this paper, we apply Leibnitz's formula and pre-Grüss inequality [3] to create several integral inequalities involving Taylor's remainder.

The present work may be considered as an continuation of the results obtained in [1] – [2].

Let $R_{n,f}(c,x)$ and $r_{n,f}\ (a,b)$ denote the $n$th Taylor's remainder of function $f$ with center $c$, and the integral Taylor's remainder, respectively, i.e.

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(n)}(c)}{n!}(x-c)^k,$$

and

$$r_{n,f}(a,b) = \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x)dx.$$

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**Lemma 1.1.** *Let $f$ be a function defined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$. Then,*

$$\int_a^b R_{n,f}(a,x)dx = \int_a^b \frac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)dx, \tag{1.1}$$

$$(-1)^{n+1} \int_a^b R_{n,f}(b,x)dx = \int_a^b \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(x)dx. \tag{1.2}$$

*Proof.* See [1]. □

**Lemma 1.2.** *Let $f$ be a function defined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$. Then*

$$r_{n,f}(a,b) = f(b) - f(a) - (b-a)f^{(1)}(a) - \cdots - \frac{(b-a)^n}{n!} f^{(n)}(a). \tag{1.3}$$

## 2. RESULTS BASED ON THE LEIBNITZ'S FORMULA

We prove the following theorem based on the Leibnitz's formula.

**Theorem 2.1.** *Let $f$ be a function defined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$. Then*

$$\left| \sum_{k=0}^{p} (-1)^k C_p^k R_{n-k,f}(a,x) \right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left| f^{(n-k)}(a) \right| \frac{(b-a)^{n-k+1}}{(n-k+1)!}, \tag{2.1}$$

$$\left| \sum_{k=0}^{p} C_p^k R_{n-k,f}(b,x) \right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left| f^{(n-k)}(b) \right| \frac{(b-a)^{n-k+1}}{(n-k+1)!}, \tag{2.2}$$

*where* $C_p^k = \frac{p!}{(p-k)!k!}$.

*Proof.* We apply the following Leibnitz's formula

$$(FG)^{(p)} = F^{(p)}G + C_p^1 F^{(p-1)}G^{(1)} + \cdots + C_p^{p-1} F^{(1)} G^{(p-1)} + FG^{(p)},$$

provided the functions $F, G \in C^p([a,b])$.

Let $F(x) = f^{(n-p+1)}(x)$, $G(x) = \frac{(b-x)^{n+1}}{(n+1)!}$. Then

$$\left( f^{(n-p+1)}(x) \frac{(b-x)^{n+1}}{(n+1)!} \right)^{(p)} = \sum_{k=0}^{p} (-1)^k C_p^k f^{(n-k+1)}(x) \frac{(b-x)^{n-k+1}}{(n-k+1)!}.$$

Integrating both sides of the preceding equation with respect to $x$ from $a$ to $b$ gives us

$$\left[ \left( f^{(n-p+1)}(x) \frac{(b-x)^{n+1}}{(n+1)!} \right)^{(p-1)} \right]_{x=a}^{x=b} = \sum_{k=0}^{p} (-1)^k C_p^k \int_a^b f^{(n-k+1)}(x) \frac{(b-x)^{n-k+1}}{(n-k+1)!} dx.$$

The integral on the right is $\int_a^b R_{n-k,f}(a,x)dx$,and to evaluate the term on the left hand side, we must again apply Leibnitz's formula, obtaining

$$-\sum_{k=0}^{p-1} (-1)^k C_{p-1}^k f^{(n-k)}(a) \frac{(b-a)^{n-k+1}}{(n-k+1)!} = \sum_{k=0}^{p} (-1)^k C_p^k \int_a^b R_{n-k,f}(a,x)dx.$$

Consequently,

$$\left| \sum_{k=0}^{p} (-1)^k C_p^k R_{n-k,f}(a,x) \right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left| f^{(n-k)}(a) \right| \frac{(b-a)^{n-k+1}}{(n-k+1)!},$$

which proves (2.1).

To prove (2.2), set $F(x) = f^{(n-p+1)}(x)$, $G(x) = \frac{(x-a)^{n+1}}{(n+1)!}$, and continue as in the proof of (2.1). □

## 3. RESULTS BASED ON THE GRÜSS TYPE INEQUALITY

We prove the following theorem based on the pre-Grüss inequality.

**Theorem 3.1.** *Let $f(x)$ be a function defined on $[a,b]$ such that $f \in C^{n+1}([a,b])$ and $m \leq f^{(n+1)}(x) \leq M$ for each $x \in [a,b]$, where $m$ and $M$ are constants. Then*

$$\left| r_{n,f}(a,b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!}(b-a)^n \right| \leq \frac{M-m}{2} \cdot \frac{n}{(2n+1)^{\frac{1}{2}}} \cdot \frac{(b-a)^{n+1}}{(n+1)!}. \tag{3.1}$$

*Proof.* We apply the following pre-Grüss inequality [3]

$$T(F,G)^2 \leq T(F,F) \cdot T(G,G), \tag{3.2}$$

where $F, G \in L_2(a,b)$ and $T(F,G)$ is the Chebyshev's functional:

$$T(F,G) = \frac{1}{b-a}\int_a^b F(x)G(x)dx - \frac{1}{b-a}\int_a^b F(x)dx \cdot \frac{1}{b-a}\int_a^b G(x)dx.$$

If there exists constants $m, M \in \mathbb{R}$ such that $m \leq F(x) \leq M$ on $[a,b]$, specially, we have [3]

$$T(F,F) \leq \frac{(M-m)^2}{4}$$

and

$$\left| \frac{1}{b-a}\int_a^b F(x)G(x)dx - \frac{1}{b-a}\int_a^b F(x)dx \cdot \frac{1}{b-a}\int_a^b G(x)dx \right| \leq \frac{1}{2}(M-m)\left[ \frac{1}{b-a}\int_a^b G^2(x)dx - \left( \frac{1}{b-a}\int_a^b G(x)dx \right)^2 \right]^{\frac{1}{2}}. \tag{3.3}$$

In formula (3.3) replacing $F(x)$ by $f^{(n+1)}(x)$, and $G(x)$ by $\frac{(b-x)^n}{n!}$, we obtain (3.1). □

**Remark 3.2.** It is possible to define the similar expression $r'_{n,f}(a,b)$ by

$$r'_{n,f}(a,b) = \int_a^b \frac{(x-a)^n}{n!} f^{(n+1)}(x)dx.$$

In exactly the same way as inequality (3.1) was obtained, one can obtain the following inequality

$$\left| r'_{n,f}(a,b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!}(b-a)^n \right| \leq \frac{M-m}{2} \cdot \frac{n}{(2n+1)^{\frac{1}{2}}} \cdot \frac{(b-a)^{n+1}}{(n+1)!}. \tag{3.4}$$

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