



# A STUDY ON STARLIKE AND CONVEX PROPERTIES FOR HYPERGEOMETRIC FUNCTIONS

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*Key words:* Starlike, Convex, Hypergeometric function, Integral operator.

*Abstract:* The objective of the present paper is to give some characterizations for a (Gaussian) hypergeometric function to be in various subclasses of starlike and convex functions. We also consider an integral operator related to the hypergeometric function.

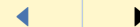
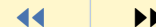
**Study on Starlike and  
Convex Properties**

A. O. Mostafa

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[Title Page](#)

[Contents](#)



Page 1 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

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# Contents

1	<a href="#">Introduction</a>	3
2	<a href="#">Main Results</a>	5
3	<a href="#">An Integral Operator</a>	13



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**Study on Starlike and  
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---

[Title Page](#)

[Contents](#)



Page 2 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

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Title Page

Contents



Page 3 of 16

Go Back

Full Screen

Close

## 1. Introduction

Let  $T$  be the class consisting of functions of the form:

$$(1.1) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

which are analytic and univalent in the open unit disc  $U = \{z : |z| < 1\}$ . Let  $T(\lambda, \alpha)$  be the subclass of  $T$  consisting of functions which satisfy the condition:

$$(1.2) \quad \operatorname{Re} \left\{ \frac{z f'(z)}{\lambda z f'(z) + (1-\lambda) f(z)} \right\} > \alpha,$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ),  $\lambda$  ( $0 \leq \lambda < 1$ ) and for all  $z \in U$ .

Also, let  $C(\lambda, \alpha)$  denote the subclass of  $T$  consisting of functions which satisfy the condition:

$$(1.3) \quad \operatorname{Re} \left\{ \frac{f'(z) + z f''(z)}{f'(z) + \lambda z f''(z)} \right\} > \alpha,$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ),  $\lambda$  ( $0 \leq \lambda < 1$ ) and for all  $z \in U$ .

From (1.2) and (1.3), we have

$$(1.4) \quad f(z) \in C(\lambda, \alpha) \Leftrightarrow z f'(z) \in T(\lambda, \alpha).$$

We note that  $T(0, \alpha) = T^*(\alpha)$ , the class of starlike functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) and  $C(0, \alpha) = C(\alpha)$ , the class of convex functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) (see Silverman [6]).

Let  $F(a, b; c; z)$  be the (Gaussian) hypergeometric function defined by

$$(1.5) \quad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

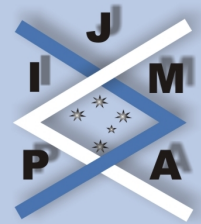
where  $c \neq 0, -1, -2, \dots$ , and  $(\theta)_n$  is the Pochhammer symbol defined by

$$(\theta)_n = \begin{cases} 1, & n = 0 \\ \theta(\theta + 1) \cdots (\theta + n - 1) & n \in N = \{1, 2, \dots\}. \end{cases}$$

We note that  $F(a, b; c; 1)$  converges for  $\operatorname{Re}(c - a - b) > 0$  and is related to the Gamma function by

$$(1.6) \quad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

Silverman [7] gave necessary and sufficient conditions for  $zF(a, b; c; z)$  to be in  $T^*(\alpha)$  and  $C(\alpha)$ , also examining a linear operator acting on hypergeometric functions. For other interesting developments on  $zF(a, b; c; z)$  in connection with various subclasses of univalent functions, the reader can refer to works of Carlson and Shaffer [2], Merkes and Scott [4], Ruscheweyh and Singh [5] and Cho et al. [3].



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 16

Go Back

Full Screen

Close



## 2. Main Results

To establish our main results, we need the following lemma due to Altintas and Owa [1].

### Lemma 2.1.

(i) A function  $f(z)$  defined by (1.1) is in the class  $T(\lambda, \alpha)$  if and only if

$$(2.1) \quad \sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha)a_n \leq 1 - \alpha.$$

(ii) A function  $f(z)$  defined by (1.1) is in the class  $C(\lambda, \alpha)$  if and only if

$$(2.2) \quad \sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha)a_n \leq 1 - \alpha.$$

### Theorem A.

(i) If  $a, b > -1, c > 0$  and  $ab < 0$ , then  $zF(a, b; c; z)$  is in  $T(\lambda, \alpha)$  if and only if

$$(2.3) \quad c > a + b + 1 - \frac{(1 - \lambda\alpha)ab}{1 - \alpha}.$$

(ii) If  $a, b > 0$  and  $c > a + b + 1$ , then  $F_1(a, b; c; z) = z[2 - F(a, b; c; z)]$  is in  $T(\lambda, \alpha)$  if and only if

$$(2.4) \quad \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left[ 1 + \frac{(1 - \lambda\alpha)ab}{(1 - \alpha)(c - a - b - 1)} \right] \leq 2.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 16

Go Back

Full Screen

Close

*Proof.* (i) Since

$$(2.5) \quad zF(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n$$

$$= z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n,$$

according to (i) of Lemma 2.1, we must show that

$$(2.6) \quad \sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left| \frac{c}{ab} \right| (1 - \alpha).$$

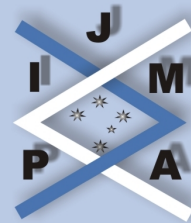
Note that the left side of (2.6) diverges if  $c < a + b + 1$ . Now

$$\sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= (1 - \lambda\alpha) \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1 - \alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

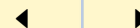
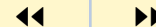
$$= (1 - \lambda\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{(1 - \alpha)c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= (1 - \lambda\alpha) \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{(1 - \alpha)c}{ab} \left[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right].$$



Title Page

Contents



Page 7 of 16

Go Back

Full Screen

Close

Hence, (2.6) is equivalent to

$$(2.7) \quad \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1-\lambda\alpha) + \frac{(1-\alpha)(c-a-b-1)}{ab} \right] \\ \leq (1-\alpha) \left[ \left| \frac{c}{ab} \right| + \frac{c}{ab} \right] = 0.$$

Thus, (2.7) is valid if and only if

$$(1-\lambda\alpha) + \frac{(1-\alpha)(c-a-b-1)}{ab} \leq 0,$$

or equivalently,

$$c \geq a + b + 1 - \frac{(1-\lambda\alpha)ab}{1-\alpha}.$$

(ii) Since

$$F_1(a, b; c; z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n,$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1 - \alpha.$$

Now,

$$(2.8) \quad \sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ = \sum_{n=2}^{\infty} [(n-1)(1-\lambda\alpha) + (1-\alpha)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 8 of 16

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\begin{aligned}
 &= (1 - \lambda\alpha) \sum_{n=1}^{\infty} n \frac{(a)_n (b)_n}{(c)_n (1)_n} + (1 - \alpha) \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \\
 &= (1 - \lambda\alpha) \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_{n-1}} + (1 - \alpha) \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n}.
 \end{aligned}$$

Noting that  $(\theta)_n = \theta((\theta + 1)_{n-1})$  then, (2.8) may be expressed as

$$\begin{aligned}
 &(1 - \lambda\alpha) \frac{ab}{c} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1} (b+1)_{n-1}}{(c+1)_{n-1} (1)_{n-1}} + (1 - \alpha) \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \\
 &= (1 - \lambda\alpha) \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} + (1 - \alpha) \left[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} - 1 \right] \\
 &= (1 - \lambda\alpha) \frac{ab}{c} \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1 - \alpha) \left[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right] \\
 &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1 - \alpha) + \frac{ab(1 - \lambda\alpha)}{(c-a-b-1)} \right] - (1 - \alpha).
 \end{aligned}$$

But this last expression is bounded above by  $1 - \alpha$  if and only if (2.4) holds.  $\square$

### Theorem B.

(i) If  $a, b > -1$ ,  $ab < 0$ , and  $c > a + b + 2$ , then  $zF(a, b; c; z)$  is in  $C(\lambda, \alpha)$  if and only if

$$\begin{aligned}
 (2.9) \quad &(1 - \lambda\alpha)(a)_2(b)_2 + (3 - 2\lambda\alpha - \alpha)ab(c - a - b - 2) \\
 &+ (1 - \alpha)(c - a - b - 2)_2 \geq 0.
 \end{aligned}$$





Title Page

Contents



Page 9 of 16

Go Back

Full Screen

Close

(ii) If  $a, b > 0$  and  $c > a + b + 2$ , then  $F_1(a, b; c; z) = z[2 - F(a, b; c; z)]$  is in  $C(\lambda, \alpha)$  if and only if

$$(2.10) \quad \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left\{ 1 + \frac{(1-\lambda\alpha)(a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2} + \left( \frac{3-2\lambda\alpha-\alpha}{1-\alpha} \right) \left( \frac{ab}{c-a-b-1} \right) \right\} \leq 2.$$

*Proof.* (i) Since  $zF(a, b; c; z)$  has the form (2.5), we see from (ii) of Lemma 2.1, that our conclusion is equivalent to

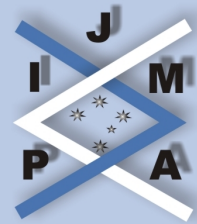
$$(2.11) \quad \sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left| \frac{c}{ab} \right| (1-\alpha).$$

Note that for  $c > a + b + 2$ , the left side of (2.11) converges. Writing

$$\begin{aligned} & (n+2)[(n+2)(1-\lambda\alpha) - \alpha(1-\lambda)] \\ &= (n+1)^2(1-\lambda\alpha) + (n+1)(2-\alpha-\lambda\alpha) + (1-\alpha), \end{aligned}$$

we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+2)[(n+2)(1-\lambda\alpha) - \alpha(1-\lambda)] \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= (1-\lambda\alpha) \sum_{n=0}^{\infty} (n+1)^2 \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &+ (2-\alpha-\lambda\alpha) \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1-\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \end{aligned}$$



Title Page

Contents



Page 10 of 16

Go Back

Full Screen

Close

$$\begin{aligned}
 &= (1 - \lambda\alpha) \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} \\
 &\quad + (2 - \alpha - \lambda\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} + (1 - \alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} \\
 &= (1 - \lambda\alpha) \sum_{n=0}^{\infty} n \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} + (3 - \alpha - 2\lambda\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} \\
 &\quad + (1 - \alpha) \sum_{n=1}^{\infty} \frac{(a+1)_{n-1} (b+1)_{n-1}}{(c+1)_{n-1} (1)_n} \\
 &= \frac{(1 - \lambda\alpha)(a+1)(b+1)}{(c+1)} \sum_{n=0}^{\infty} \frac{(a+2)_n (b+2)_n}{(c+2)_n (1)_n} \\
 &\quad + (3 - \alpha - 2\lambda\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} + (1 - \alpha) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \\
 &= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \left[ - (1 - \lambda\alpha)(a+1)(b+1) \right. \\
 &\quad \left. + (3 - \alpha - 2\lambda\alpha)(c-a-b-2) + \frac{(1-\alpha)}{ab}(c-a-b-2)_2 \right] - \frac{(1-\alpha)c}{ab}.
 \end{aligned}$$

This last expression is bounded above by  $\left| \frac{c}{ab} \right| (1 - \alpha)$  if and only if

$$(1 - \lambda\alpha)(a+1)(b+1) + (3 - \alpha - 2\lambda\alpha)(c-a-b-2) + \frac{(1-\alpha)}{ab}(c-a-b-2)_2 \leq 0,$$

which is equivalent to (2.9).



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 16

Go Back

Full Screen

Close

(ii) In view of (ii) of Lemma 2.1, we need to show that

$$\sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (1 - \alpha).$$

Now

$$\begin{aligned} (2.12) \quad & \sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= \sum_{n=0}^{\infty} (n+2)[(n+2)(1 - \lambda\alpha) - \alpha(1 - \lambda)] \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &= (1 - \lambda\alpha) \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - \alpha(1 - \lambda) \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \end{aligned}$$

Writing  $(n+2) = (n+1) + 1$ , we have

$$\begin{aligned} (2.13) \quad & \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} = \sum_{n=0}^{\infty} (n+1) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \end{aligned}$$

and

$$\begin{aligned} (2.14) \quad & \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &= \sum_{n=0}^{\infty} (n+1) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + 2 \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 16

Go Back

Full Screen

Close

$$= \sum_{n=1}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + 3 \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

Substituting (2.13) and (2.14) into the right side of (2.12), yields

$$(2.15) \quad (1 - \lambda\alpha) \sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + (3 - 2\lambda\alpha - \alpha) \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + (1 - \alpha) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

Since  $(a)_{n+k} = (a)_k(a+k)_n$ , we may write (2.15) as

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ \frac{(1-\lambda\alpha)(a)_2(b)_2}{(c-a-b-2)_2} + \frac{(3-2\lambda\alpha-\alpha)ab}{(c-a-b-1)} + (1-\alpha) \right] - (1-\alpha).$$

By a simplification, we see that the last expression is bounded above by  $(1 - \alpha)$  if and only if (2.10) holds.  $\square$

Putting  $\lambda = 0$  in (i) of Theorem B, we have:

**Corollary 2.2.** *If  $a, b > -1$ ,  $ab < 0$ , and  $c > a + b + 2$ , then  $zF(a, b; c; z)$  is in  $C(\alpha)$  if and only if*

$$(a)_2(b)_2 + (3 - \alpha)ab(c - a - b - 2) + (1 - \alpha)(c - a - b - 2)_2 \geq 0.$$

*Remark 1.* Corollary 2.2, corrects the result obtained by Silverman [7, Theorem 4].



Title Page

Contents



Page 13 of 16

Go Back

Full Screen

Close

### 3. An Integral Operator

In the theorems below, we obtain similar results in connection with a particular integral operator  $G(a, b; c; z)$  acting on  $F(a, b; c; z)$  as follows:

$$(3.1) \quad G(a, b; c; z) = \int_0^z F(a, b; c; t) dt.$$

**Theorem C.** *Let  $a, b > -1, ab < 0$  and  $c > \max\{0, a + b\}$ . Then  $G(a, b; c; z)$  defined by (3.1) is in  $T(\lambda, \alpha)$  if and only if*

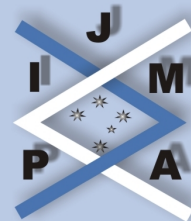
$$(3.2) \quad \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ \frac{(1-\lambda\alpha)}{ab} - \frac{\alpha(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2} \right] + \frac{\alpha(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2} \leq 0.$$

*Proof.* Since

$$G(a, b; c; z) = z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} z^n,$$

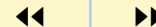
by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} (n - \lambda\alpha n - \alpha + \lambda\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \leq \left| \frac{c}{ab} \right| (1 - \alpha).$$



Title Page

Contents



Page 14 of 16

Go Back

Full Screen

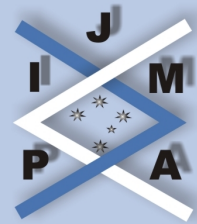
Close

Now

$$\begin{aligned}
 & \sum_{n=2}^{\infty} [n(1-\lambda\alpha) - \alpha(1-\lambda)] \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \\
 &= (1-\lambda\alpha) \sum_{n=0}^{\infty} (n+2) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} - \alpha(1-\lambda) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\
 &= (1-\lambda\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} - \alpha(1-\lambda) \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n+1}} \\
 &= (1-\lambda\alpha) \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n} - \alpha(1-\lambda) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} \\
 &= (1-\lambda\alpha) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} - \alpha(1-\lambda) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} \\
 &= (1-\lambda\alpha) \frac{c}{ab} \left[ \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} - 1 \right] \\
 &\quad - \alpha(1-\lambda) \frac{(c-1)_2}{(a-1)_2(b-1)_2} \sum_{n=2}^{\infty} \frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n} \\
 &= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ \frac{(1-\lambda\alpha)}{ab} - \frac{\alpha(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2} \right] \\
 &\quad + \frac{\alpha(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2} - \frac{(1-\alpha)c}{ab},
 \end{aligned}$$

which is bounded above by  $(1-\alpha) \left| \frac{c}{ab} \right|$  if and only if (3.2) holds.  $\square$

Now, we observe that  $G(a, b; c; z) \in C(\lambda, \alpha)$  if and only if  $zF(a, b; c; z) \in$



Title Page

Contents



Page 15 of 16

Go Back

Full Screen

Close

$T(\lambda, \alpha)$ . Thus any result of functions belonging to the class  $T(\lambda, \alpha)$  about  $zF(a, b; c; z)$  leads to that of functions belonging to the class  $C(\lambda, \alpha)$ . Hence we obtain the following analogous result to Theorem A.

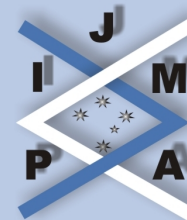
**Theorem 3.1.** *Let  $a, b > -1, ab < 0$  and  $c > a + b + 2$ . Then  $G(a, b; c; z)$  defined by (3.1) is in  $C(\lambda, \alpha)$  if and only if*

$$c > a + b + 1 - \frac{(1 - \lambda\alpha)ab}{1 - \alpha}.$$

*Remark 2.* Putting  $\lambda = 0$  in the above results, we obtain the results of Silverman [7].

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Title Page

Contents



Page 16 of 16

Go Back

Full Screen

Close

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