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SOME REMARKS ON LOWER BOUNDS OF CHEBYSHEV'S TYPE FOR HALF-LINES

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ABSTRACT. We prove that for any r.v. X such that $E\{X\}=0,\ E\{X^2\}=1,$ and $E\{X^4\}=\mu$, and for any $\varepsilon \geq 0$

$$P(X \geq \varepsilon) \geq \frac{K_0}{\mu} - \frac{K_1}{\sqrt{\mu}}\varepsilon + \frac{K_2}{\mu\sqrt{\mu}}\varepsilon,$$

where absolute constants $K_0 = 2\sqrt{3}-3 \approx 0.464,\ K_1 = 1.397,$ and $K_2 = 0.0231.$ The constant K_0 is sharp for $\mu \geq \frac{3}{\sqrt{3}+1} \approx 1.09$. Some other bounds and examples are given.

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1. INTRODUCTION AND RESULTS

Let X be a r.v. such that $E\{X\}=0,\ E\{X^2\}=1,\ E\{X^4\}=\mu$. It is well known (see, e.g., [4, Chapter XII, 3]) that for any $\varepsilon \in [0,1]$

$$P(|X| > \varepsilon) \geq \frac{(1-\varepsilon^2)^2}{\mu - 1 + (1-\varepsilon^2)^2} \geq \frac{(1-\varepsilon^2)^2}{\mu}.$$

The first inequality is sharp, the second is somewhat simpler, and is used, for example, for proving the Paley-Zygmund inequality (see, e.g., [3]). (There is a reason to involve, not the third absolute, but the fourth moment (see, e.g., [4]): the highest moment should be absolute, and the third absolute moment is hard to calculate, for example, when X is the sum of r.v.'s.)

Although there has been a great deal of interest in obtaining bounds of such a type, we have been unable to find a handy and useful lower bound for the "one-sided" probability $P(X > \varepsilon)$

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in the literature. Possibly it is because such a bound, as will be seen, is not so simple in proof, and can be meaningful only for sufficiently small ε. However we suspect that such results may exist.

A need for a convenient lower bound for the probability mentioned may arise in many problems. We encountered such a need recently in [1], in the study of the dimension of the sets of convergence for some random series.

The main result of this note is

Proposition 1.1. *For any r.v. X described above, and for any $\varepsilon \geq 0$*

$$P(X \geq \varepsilon) \geq \frac{K_0}{\mu} - \frac{K_1}{\sqrt{\mu}}\varepsilon + \frac{K_2}{\mu\sqrt{\mu}}\varepsilon, \tag{1.1}$$

where absolute constants $K_0 = 2\sqrt{3} - 3 \approx 0.464,\ K_1 = 1.397,\ K_2 = 0.0231.$

In particular,

$$P(X > 0) \geq \frac{K_0}{\mu}. \tag{1.2}$$

We show below that the last bound, and hence the constant K_0 in (1.1), is sharp if

$$\mu \geq \frac{3}{\sqrt{3}+1} \approx 1.098. \tag{1.3}$$

When $\mu \leq \frac{3}{\sqrt{3}+1}$, the sharp bound, as will be shown, is

$$P(X > 0) \geq \frac{2}{3 + \mu + \sqrt{(1+\mu)^2 - 4}}. \tag{1.4}$$

The r.-h.s of (1.2) is equal to the r.-h.s of (1.4) for $\mu = \frac{3}{\sqrt{3}+1}$, and is less for all other μ's. For $\mu \leq \frac{3}{\sqrt{3}+1}$ we can choose (1.4), while for $\mu > \frac{3}{\sqrt{3}+1}$ the proper bound is (1.2). We do not obtain here the counterpart of (1.1) with a sharp constant for $\mu < \frac{3}{\sqrt{3}+1}$: first, this case is rather narrow: $1 \leq \mu < 1.1$; second, (1.1) which is true for all μ, may serve well for this range of μ as well: say, for $\mu = 1$ the sharp bound is certainly $\frac{1}{2}$, which does not differ much from 0.464.

Since K_2 is small and the denominator in the third term of (1.1) is larger than the denominator in the second term, practically we can restrict ourselves to the bound

$$P(X \geq \varepsilon) \geq \frac{K_0}{\mu} - \frac{K_1}{\sqrt{\mu}}\varepsilon. \tag{1.5}$$

This bound is meaningful if

$$\varepsilon \leq \frac{K_3}{\sqrt{\mu}}, \tag{1.6}$$

where $K_3 = \frac{K_0}{K_1} \geq 0.332$. The last constant is not sharp. However the restriction of type (1.6) with some constant is necessary for the bound for $P(X \geq \varepsilon)$ to be meaningful. For example, as will be shown, for any $\mu \geq 1$ there exists a r.v. $\dot{X}$ with the above moment conditions, such that

$$P\left(X > \frac{1}{\sqrt{\mu}}\right) = 0. \tag{1.7}$$

Certainly, this does not mean that K_3 is equal to one. In particular, we will see below that there exists $\mu > 1$ and a r.v. X with the same moment conditions such that

$$P\left(X > \frac{\sqrt{3}}{2\sqrt{\mu}}\right) = 0. \tag{1.8}$$

This means that K_3 should not exceed $\frac{\sqrt{3}}{2} \approx 0.866$.

2. EXAMPLES

(1) The first example is simple and could be used in teaching. Let $z \geq 0$, and

$$X = \begin{cases} \sqrt{z} & \text{with probability } \frac{1}{z+1}, \\ \frac{-1}{\sqrt{z}} & \text{with probability } \frac{z}{z+1}. \end{cases}$$

For small (or for large) z the distribution is "strongly asymmetric", but $E\{X\} = 0,\ E\{X^2\} = 1$, and

$$E\{X^4\} = z + \frac{1}{z} - 1,$$

which can be equal to any number ≥ 1. Setting $z + z^{-1} - 1 = \mu$, we get $z_1 = z(\mu) = \left(1 + \mu + \sqrt{(1+\mu)^2 - 4}\right) \Big/ 2,\quad z_2 = \frac{1}{z(\mu)}$. It is easy to see that $\mu \leq z(\mu) \leq 1 + \mu$, and hence for $z = z_2$ (1.7) is true. Straightforward calculations show that $\min_{\mu \geq 1} \sqrt{\frac{\mu}{z(\mu)}} = \frac{\sqrt{3}}{2}$, and is attained at $\mu = \frac{3}{2}$. So, for this μ (1.8) holds.

(2) As is known, and as will be seen in Section 3, the extreme distribution in our problem is that concentrated at just three points. It is easy to realize also (see, for example, the next section) that for the case $\varepsilon = 0$ one of these points is zero. Restricting ourselves for a while to this case, consider

$$X = \begin{cases} \sqrt{\mu}a & \text{with probability } u, \\ 0 & \text{with probability } 1 - u - v, \\ -\sqrt{\mu}b & \text{with probability } v, \end{cases}$$

where a, b, u, v are positive numbers. For $E\{X\} = 0,\ E\{X^2\} = 1,\ E\{X^4\} = \mu$ one should have

$$au = bv, \quad a^2u + b^2v = \frac{1}{\mu}, \quad a^4u + b^4v = \frac{1}{\mu}, \tag{2.1}$$

and

$$0 \leq u + v \leq 1. \tag{2.2}$$

It is easy to check that solutions to (2.1) may be represented as

$$u = \frac{1}{\mu}\frac{x^2 - x + 1}{x + 1}, \quad v = \frac{1}{\mu}\frac{x^2 - x + 1}{x(x+1)},$$
$$a^2 = \frac{1}{x^2 - x + 1}, \qquad b^2 = \frac{x^2}{x^2 - x + 1},$$

where $x > 0$ (one can set $x = \frac{u}{v}$, and solve (2.1) directly). For example, setting $x = 1$, we have $u = \frac{1}{2\mu},\ \ v = \frac{1}{2\mu},\ \ a = 1,\ \ b = 1$, and

$$X = \begin{cases} \sqrt{\mu} & \text{with probability } \frac{1}{2\mu}, \\ 0 & \text{with probability } 1 - \frac{1}{\mu}, \\ -\sqrt{\mu} & \text{with probability } \frac{1}{2\mu}, \end{cases}$$

which certainly is not the extreme case.

To check (2.2), we note that $u + v = \frac{1}{\mu}(x + x^{-1} - 1)$.

Hence for (2.2) to be true, we should have $\frac{1}{z(\mu)} \le x \le z(\mu)$, where $z(\mu)$ is the same as above. In this case $P(X > 0) = \frac{1}{\mu} \cdot \frac{x^2-x+1}{x+1}$. The minimum of the last expression is attained at $x^* = \sqrt{3} - 1$, and in this case $P(X > 0) = \frac{1}{\mu}(2\sqrt{3} - 3) = \frac{K_0}{\mu}$. Hence, if $x^* \ge \frac{1}{z(\mu)}$, the bound (1.2) is sharp.

It is straightforward to verify that $x^* \ge \frac{1}{z(\mu)}$ iff $\mu \ge \frac{3}{\sqrt{3}+1} \approx 1.098$. If $\mu \le \frac{3}{\sqrt{3}+1}$, then the minimum of $P(X > 0)$ is attained at $x = z_2(\mu) = \frac{1}{z(\mu)}$. In this case $P(X = 0) = 0$, $x^2 - x + 1 = \mu x$, and

$$P(X > 0) = \frac{z_2(\mu)}{1 + z_2(\mu)} = \frac{2}{3 + \mu + \sqrt{(1+\mu)^2 - 4}}. \tag{2.3}$$

Thus, for $\mu \le \frac{3}{\sqrt{3}+1}$ we attain the bound (1.4). For $\mu > \frac{3}{\sqrt{3}+1}$ the r.-h.s. of (2.3) is greater than $\frac{K_0}{\mu}$.

3. Proof of Proposition 1.1

We consider an appropriate *upper* bound for $P(X \le \varepsilon)$ following the well known method based on the use of polynomials of a certain order (see, e.g., [2], [4]). In our case these are polynomials $g(x) = a_0 + a_1 x + a_2 x^2 + a_4 x^4$ such that for all x

$$I_{[-\infty,\varepsilon]}(x) \le g(x). \tag{3.1}$$

Then for each such polynomial

$$P(X \le \varepsilon) \le a_0 + a_2 + a_4 \mu. \tag{3.2}$$

Minimizing the right-hand side over all polynomials satisfying (3.1), one obtains a sharp upper bound for the left-hand side; see again, e.g., [2], [4]. Considering a smaller class of polynomials with the same property one would get just an upper bound. Let $a, b, k \ge 0$, and

$$g(x) = b(x-a)^2\left[(x+a)^2 + ka^2\right] = b\left[(x^2-a^2)^2 + ka^2(x-a)^2\right].$$

(In this case the coefficient for x^3 vanishes). It is easy to check that for $k < \frac{1}{2}$ the function g has a local maximum at the point $x_1 = a(\sqrt{1-2k} - 1)/2$, and local minima at the points a and $x_2 = -a(1 + \sqrt{1-2k})/2$.

Let $\nu = \frac{\varepsilon}{a}$. Then $g(\varepsilon) = ba^4 l(\nu, k)$, where $l(\nu, k) = (1-\nu^2)^2 + k(1-\nu)^2$. Assume that

$$\nu \le s, \tag{3.3}$$

where the number $s < 1$ will be specified later. We have

$$\begin{aligned} l(\nu,k) &= 1 + k - 2\nu^2 + \nu^4 - 2k\nu + k\nu^2 \\ &\le 1 + k - 2k\nu - (2 - s^2 - k)\nu^2 \\ &\le 1 + k - 2k\nu \le 1 + k, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} l(\nu,k) &= 1 + k - 2\nu^2 + \nu^4 - 2k\nu + k\nu^2 \\ &\ge 1 + k - 2k\nu - (2-k)\nu^2 \\ &\ge 1 + k - 2k\nu - (2-k)s\nu \\ &= 1 + k - (2k + (2-k)s)\nu. \end{aligned} \tag{3.5}$$

Furthermore, $g(x_2) = ba^4 q(k)$, where

$$q(k) = 4^{-1}\left[4^{-1}\left(\left(1+\sqrt{1-2k}\right)^2 - 4\right)^2 + k\left(3+\sqrt{1-2k}\right)^2\right]$$
$$= 4^{-1}\left[2+10k-k^2-(2-4k)\sqrt{1-2k}\right].$$

The function $\Delta(k) := q(k) - 1 - k$ is increasing; $\Delta(.5) \approx .1875$, and $\Delta(k_0) = 0$ for $k_0 = 6\sqrt{3} - 10 \approx .392$, which one can verify by direct calculations. Thus, for any $k \in \left[k_0, \frac{1}{2}\right]$ it is true that $q(k) \geq 1 + k > l(\nu, k)$ and if $g(\varepsilon) \geq 1$, then $g(x) \geq g(x_2) \geq g(\varepsilon) \geq 1$ for all $x \leq \varepsilon$.

We set also $b = 1/a^4 l(\nu, k)$ for $g(\varepsilon) = 1$.

Consider a r.v. X such that $EX = 0$, $EX^2 = 1$, $EX^4 = \mu$. Then

$$\begin{aligned} P(X \leq \varepsilon) &\leq E\{g(X)\} \\ &= bE\left\{X^4 - 2X^2a^2 + a^4 + ka^2(X^2 - 2aX + a^2)\right\} \\ &= b\left[(1+k)a^4 - (2-k)a^2 + \mu\right] \\ &= \frac{1+k}{l(\nu,k)} - \frac{2-k}{l(\nu,k)a^2} + \frac{\mu}{l(\nu,k)a^4}. \end{aligned}$$

So, in view of (3.3), (3.4) and (3.5)

$$P(X \leq \varepsilon) \leq \frac{1+k}{1+k-(2k+(2-k)s)\nu} - \frac{2-k}{(1+k-2k\nu)a^2} + \frac{\mu}{(1+k-(2k+(2-k)s)\nu)a^4}.$$

To avoid cumbersome calculations we minimize the last expression in a, not taking into account for a while that ν, as a matter of fact, depends on a. That is, we set

$$a^2 = \frac{2\mu(1+k-2k\nu)}{(1+k-(2k+(2-k)s)\nu)(2-k)}, \tag{3.6}$$

which implies that

$$\begin{aligned} P(X \leq \varepsilon) &\leq \frac{1+k}{1+k-(2k+(2-k)s)\nu} - \frac{1}{4}\cdot\frac{(2-k)^2(1+k-(2k+(2-k)s)\nu)}{(1+k-2k\nu)^2}\cdot\frac{1}{\mu} \\ &= 1 + \frac{(2k+(2-k)s)\nu}{1+k-((2-s)k+2s)\nu} - \frac{(2-k)^2}{4\mu(1+k)} \\ &\qquad + \frac{(2-k)^2}{4\mu(1+k)}\left(1 - \frac{(1+k)(1+k-(2k+(2-k)s)\nu)}{(1+k-2k\nu)^2}\right). \end{aligned}$$

It is easy to check that the expression in the last brackets does not exceed $\nu[-2(1+k)k + 4k^2s + (1+k)(2-k)s]/(1+k-2k\nu)^2$.

Note also that, for a chosen

$$\frac{2\mu(1+k-2ks)}{(1+k)(2-k)} \leq a^2 \leq \frac{2\mu(1+k)}{(1+k-(2ks+(2-k)s^2)(2-k)}.$$

From this it follows that

$$\begin{aligned} P(X \leq \varepsilon) \leq 1 - \frac{(2-k)^2}{4\mu(1+k)} &+ \frac{(2k+(2-k)s)}{1+k-(2k+(2-k)s)s}\cdot\frac{\varepsilon}{a} \\ &+ \frac{(2-k)^2}{4\mu(1+k)}\cdot\frac{(1+k)(2-k)s+4k^2s-2(1+k)k}{(1+k-2k\nu)^2}\cdot\frac{\varepsilon}{a}. \end{aligned}$$

We now set $k = k_0 = 6\sqrt{3} - 10$. Then

$$a^2 \le \frac{2.7847\mu}{(1.3923 - (0.7847s + 1.6077s^2))1.6076} = \frac{1.7322\mu}{1.3923 - 0.7847s - 1.6077s^2},$$

and

$$a^2 \ge \frac{2\mu(1.3923 - 0.7847s)}{1.3924 \cdot 1.6077} \ge 0.8934\mu(1.3923 - 0.7847s). \tag{3.7}$$

Furthermore, let $K_0 = (2 - k_0)^2/4(1 + k_0) = 2\sqrt{3} - 3$. We consider now $s \le .38$, which implies that $(1 + k)(2 - k)s + 4k^2 s - 2(1 + k)k \le 2.90s - 1.09 < 0$. Thus

$$\begin{aligned} P\left(X \le \varepsilon\right) &\le 1 - \frac{K_0}{\mu} + \frac{0.7847 + 1.6077s}{1.3923 - 0.7847s - 1.6077s^2} \cdot \frac{\varepsilon}{\sqrt{0.8934\mu(1.3923 - 0.7847s)}} \\ &\qquad + \frac{0.4641}{\mu} \cdot \frac{2.8540s - 1.0924}{1.9386} \cdot \frac{\sqrt{1.3923 - 0.7847s - 1.6077s^2}}{\sqrt{1.7322\mu}} \varepsilon \\ &\le 1 - \frac{K_0}{\mu} + \frac{(1.0570)(0.7847 + 1.6077s)}{1.3923 - 0.7847s - 1.6077s^2} \cdot \frac{\varepsilon}{\sqrt{\mu(1.3923 - 0.7847s)}} \\ &\qquad - 0.1818 \cdot (1.0924 - 2.8540s)\sqrt{1.3923 - 0.7847s - 1.6077s^2}\, \frac{\varepsilon}{\mu\sqrt{\mu}} \\ &= 1 - \frac{K_0}{\mu} + \frac{C_1(s)\varepsilon}{\sqrt{\mu}} - \frac{C_2(s)\varepsilon}{\mu\sqrt{\mu}}. \end{aligned} \tag{3.8}$$

On the other hand, the requirement (3.3) means $\varepsilon \le as$, which is true if

$$\varepsilon \le C_3(s)\sqrt{\mu} = s\sqrt{1.2438 - 0.7011s}\sqrt{\mu} \le s\sqrt{0.8934\mu(1.3923 - 0.7847s)}\ .$$

(see (3.7)).

We choose s for which

$$C_3(s) \ge \frac{K_0}{C_1(s) - C_2(s)}.$$

The bound (3.8) is meaningful if

$$\varepsilon \le \frac{K_0\sqrt{\mu}}{\mu C_1(s) - C_2(s)} \le \frac{K_0\sqrt{\mu}}{C_1(s) - C_2(s)} \le C_3(s)\sqrt{\mu}.$$

Calculations show that we can choose $s = 0.3375$. In this case $C_3(s) \ge 0.3382$, and $(K_0/(C_1(s) - C_2(s)) \le 0.33793,\ \ C_1(s) \le 1.3965$, $C_2(s) \ge 0.0231$. □

Remark 3.1. The proof above is sharp in the case when the infimum of the r.-h.s. of (3.2) is attained at some polynomial. This is not the case when $\mu < \frac{3}{\sqrt{3}+1}$: in this situation the infimum is not attained, and, to make the proof sharp, one should consider a sequence of polynomials $g_n(x)$ such that $g_n(\varepsilon) \to \infty$. It is easy to see this, considering, for example, the extreme case $\mu = 1$. We skipped such calculations.

Remark 3.2. The representation for polynomials could be different. For example, to evaluate just $P(X > 0)$ it is convenient to consider polynomials of the type $g(x) = 1 + kx(x - u)^2(x + 2u)$, where k, u are constants, which practically immediately would bring us to (1.2). To get a bound for $P(X > \varepsilon)$, one could consider the same polynomial replacing the r.v. X by $X - \varepsilon$ (see, e.g., such a sort of reasoning in [2]). However in our calculations it led us to worse constants K_2 and K_3.

Remark 3.3. The same concerns the direct way based on considering distributions concentrated at three points (as we did in the previous section). The start is easy but in our calculations it led to worse constants.

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