

Journal of Inequalities in Pure and Applied Mathematics

A REFINEMENT OF VAN DER CORPUT'S INEQUALITY

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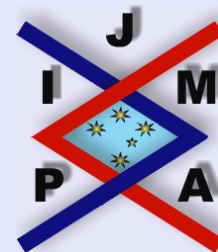
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©2000 Victoria University
ISSN (electronic): 1443-5756
136-06



volume 7, issue 4, article 127,
2006.

*Received 08 May, 2006;
accepted 10 August, 2006.*

Communicated by: B. Yang

Abstract

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Abstract

In this note, a refinement of van der Corput's inequality is given.

2000 Mathematics Subject Classification: Primary 26D15.

Key words: Refinement, van der Corput's inequality, Harmonic number.

The authors were supported in part by the Science Foundation of the Project for Fostering Innovation Talents at Universities of Henan Province, China

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1. Introduction

Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} (n+1)a_n < \infty$, and $S_n = \sum_{m=1}^n \frac{1}{m}$, the harmonic number. Then van der Corput's inequality [5] states that

$$(1.1) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} (n+1)a_n,$$

where $\gamma = 0.57721566 \dots$ stands for Euler-Mascheroni's constant. The factor $e^{1+\gamma}$ in (1.1) is the best possible.

In 2003, Hu in [3] gave a strengthened version of (1.1) by

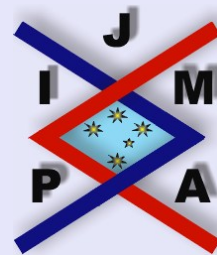
$$(1.2) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{4} \right) a_n.$$

Recently, Yang in [7] obtained a better result than Hu's inequality (1.2) as

$$(1.3) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{3} \right) a_n.$$

Moreover, he also extended (1.1) in [7] as follows

$$(1.4) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/(k+\beta)} \right)^{\frac{1}{S_n(\beta)}} < e^{1+\gamma(\beta)} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} + \beta \right) a_n,$$



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where $\beta \in (-1, \infty)$, $S_n(\beta) = \sum_{k=1}^n \frac{1}{k+\beta}$, and

$$\gamma(\beta) = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k+\beta} - \ln(n+\beta) \right].$$

Applying $\beta = 0$ in (1.4) leads to

$$(1.5) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} \right) a_n,$$

which improved inequality (1.1) clearly, but is not more accurate than (1.2) and (1.3).

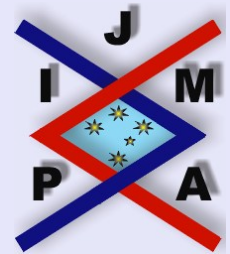
In [1], among other things, the authors established a sharper inequality than (1.1), (1.2), (1.3) and (1.5) as follows

$$(1.6) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{3n - 1/4} \right) a_n.$$

The purpose of this note is to refine further inequality (1.6). Our main result is the following.

Theorem 1.1. For $n \in \mathbb{N}$, let $S_n = \sum_{m=1}^n \frac{1}{m}$, the harmonic number. If $a_n \geq 0$ for $n \in \mathbb{N}$ and

$$0 < \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6} \right) a_n < \infty,$$



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then

$$(1.7) \quad \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6} \right) a_n,$$

where $\gamma = 0.57721566\dots$ is Euler-Mascheroni's constant.

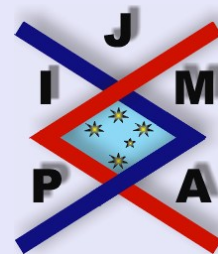
Remark 1. Let

$$A_n = e^{1+\gamma} e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6} \right)$$

for $n \in \mathbb{N}$. Numerical computation shows $A_1 = 4.40\dots < e^{1+\gamma} \left(1 - \frac{\ln 1}{3-1/4} \right) = 4.84\dots$ and $A_2 = 7.99\dots < e^{1+\gamma} \left(2 - \frac{2\ln 2}{6-1/4} \right) = 8.51\dots$ and $A_3 = 11.95\dots < e^{1+\gamma} \left(3 - \frac{3\ln 2}{9-1/4} \right) = 12.70\dots$. When $n \geq 4$, inequality $2n + \frac{11}{6} + \ln n < 3n - \frac{1}{4}$ is valid, which can be rearranged as

$$1 - \frac{\ln n}{2n + \ln n + 11/6} \leq 1 - \frac{\ln n}{3n - 1/4}.$$

This implies that inequality (1.7) is a refinement of (1.6).



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2. Lemmas

In order to prove our main result, some lemmas are necessary.

Lemma 2.1 ([4]). For $n \in \mathbb{N}$,

$$(2.1) \quad \frac{1}{2n+1/(1-\gamma)-2} < S_n - \ln n - \gamma < \frac{1}{2n+1/3}.$$

The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ in (2.1) are the best possible.

Lemma 2.2 ([2, 6]). If $x > 0$, then

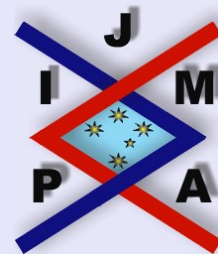
$$(2.2) \quad \left(1 + \frac{1}{x}\right)^x < e \left(1 - \frac{1}{2x+11/6}\right).$$

Lemma 2.3. For $n \in \mathbb{N}$,

$$(2.3) \quad B_n \triangleq \left[\frac{(n+1)S_{n+1}}{nS_n}\right]^{nS_n} < e^{1+\gamma} e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6}\right).$$

Proof. By virtue of Lemma 2.2, it follows that

$$\begin{aligned} \left[\frac{(n+1)S_{n+1}}{nS_n}\right]^{\frac{nS_n}{S_{n+1}}} &= \left(1 + \frac{S_n+1}{nS_n}\right)^{\frac{nS_n}{S_{n+1}}} \\ &< e \left[1 - \frac{S_n+1}{2nS_n+11(S_n+1)/6}\right] < e \left(1 - \frac{1}{2n+11/6}\right). \end{aligned}$$



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Applying Lemma 2.1 yields

$$(2.4) \quad B_n < \left[e \left(1 - \frac{1}{2n + 11/6} \right) \right]^{S_n + 1} \\ < e^{1 + \frac{1}{2n + 1/3} + \gamma + \ln n} \left(1 - \frac{1}{2n + 11/6} \right)^{1 + \frac{1}{2n + 1/3} + \gamma + \ln n}.$$

Taking advantage of inequalities $(1 - \frac{1}{x})^{-x} > e$ for $x > 1$ and $e^{-x} \leq \frac{1}{1+x}$ for $x > -1$ leads to

$$(2.5) \quad \left(1 - \frac{1}{2n + 11/6} \right)^{\ln n} \leq \exp \left(-\frac{\ln n}{2n + 11/6} \right) \leq \frac{2n + 11/6}{2n + 11/6 + \ln n}$$

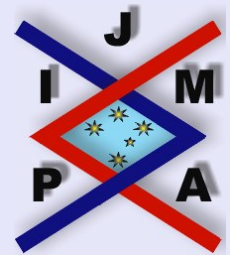
and

$$(2.6) \quad \left(1 - \frac{1}{2n + 11/6} \right)^{\frac{1}{2n + 1/3} + 1 + \gamma} \exp \left(\frac{1}{2n + 1/3} \right) \\ < \exp \left[\frac{1}{2n + 1/3} - \frac{1 + \gamma}{2n + 11/6} - \frac{1}{(2n + 11/6)(2n + 1/3)} \right] \\ = \exp \left[-\frac{6(6n + 1)\gamma - 9}{(6n + 1)(12n + 11)} \right].$$

Combination of (2.4), (2.5), (2.6) gives

$$(2.7) \quad B_n < e^{1 + \gamma - \frac{6(6n + 1)\gamma - 9}{(6n + 1)(12n + 11)}} \left(n - \frac{n \ln n}{2n + \ln n + 11/6} \right).$$

Lemma 2.3 is proved. □



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3. Proof of Theorem 1.1

For $n \in \mathbb{N}$ and $1 \leq k \leq n$, let

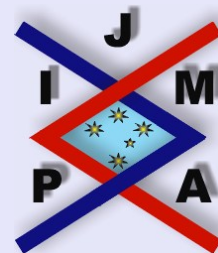
$$(3.1) \quad c_k = \frac{[(k+1)S_{k+1}]^{kS_k}}{(kS_k)^{kS_{k-1}}}$$

with assumption $S_0 = 0$, then

$$(3.2) \quad \left(\prod_{k=1}^n c_k^{1/k} \right)^{-\frac{1}{S_n}} = \frac{1}{(n+1)S_{n+1}}.$$

By using the discrete weighted arithmetic-geometric mean inequality and interchanging the order of summations,

$$(3.3) \quad \begin{aligned} \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{\frac{1}{S_n}} &= \sum_{n=1}^{\infty} \left[\prod_{k=1}^n (c_k a_k)^{1/k} \right]^{\frac{1}{S_n}} \left(\prod_{k=1}^n c_k^{1/k} \right)^{-\frac{1}{S_n}} \\ &\leq \sum_{n=1}^{\infty} \left(\prod_{k=1}^n c_k^{1/k} \right)^{-\frac{1}{S_n}} \frac{1}{S_n} \sum_{k=1}^n \frac{c_k a_k}{k} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{(n+1)S_{n+1}S_n} \sum_{k=1}^n \frac{c_k a_k}{k} \\ &= \sum_{k=1}^{\infty} \frac{c_k a_k}{k} \sum_{n=k}^{\infty} \frac{1}{(n+1)S_n S_{n+1}} \\ &= \sum_{k=1}^{\infty} \frac{c_k a_k}{k} \sum_{n=k}^{\infty} \left(\frac{1}{S_n} - \frac{1}{S_{n+1}} \right) \end{aligned}$$



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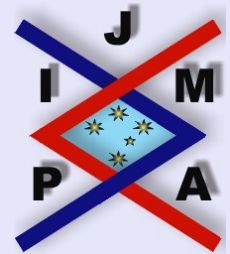
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$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{c_k a_k}{k S_k} \\
&= \sum_{k=1}^{\infty} \left[\frac{(k+1) S_{k+1}}{k S_k} \right]^{k S_k} a_k = \sum_{n=1}^{\infty} B_n a_n.
\end{aligned}$$

Substituting (2.7) into (3.3) leads to (1.7). The proof of Theorem 1.1 is complete.



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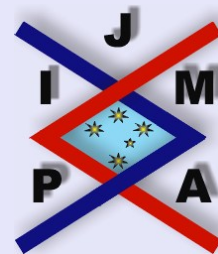
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