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MINTY VARIATIONAL INEQUALITIES AND MONOTONE TRAJECTORIES OF DIFFERENTIAL INCLUSIONS

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ABSTRACT. In [8] the notion of "projected differential equation" has been introduced and the stability of solutions has been studied by means of Stampacchia type variational inequalities. More recently, in [20], Minty variational inequalities have been involved in the study of properties of the trajectories of such a projected differential equation.
We consider classical generalizations of both problems, namely projected differential inclusions and variational inequalities with point to set operators, and we extend results stated in [20] to this setting. Moreover, we also apply the results to describe the convergence of the trajectories of a generalized gradient inclusion method.

Key words and phrases: Minty variational inequalities, differential inclusions, monotone trajectories, slow solutions.

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1. INTRODUCTION

The relations of Minty and Stampacchia Variational Inequalities [21] with differentiable optimization problems have been widely studied. Basically, it has been proved that the Stampacchia Variational Inequality (for short, SVI) is a necessary condition for optimality (see e.g. [14]), while the Minty Variational Inequality (for short, MVI) is a sufficient one (see e.g. [7, 11, 15]). Generalizations of SVI and MVI to point to set maps have been introduced (see e.g. [4, 9]) and the previous results have been proved also for non differentiable optimization problems (see e.g. [5]).

On the other hand, Dynamical Systems (for short, DS) are a classical tool for dealing with a wide range both of real and mathematical problems. Recently, the existence and stability of

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equilibria of a (projected) DS have been characterized by means of variational inequalities. In this context it has been proved that existence of a solution of SVI is equivalent to existence of an equilibrium, while MVI ensures the stability of equilibria (see [8, 20]).

The latter results proved to be useful in deriving a wide variety of applications and a deeper insight on the dynamic of the adjustment towards an equilibrium. Basically, variational inequalities are used to model static equilibria of several economies, such as Cournot oligopoly, spatial oligopoly, general economic equilibrium and so on [18], while dynamical systems (or more realistically differential inclusions) are used to describe the path to equilibrium, starting from a given state of the world (see e.g. [10]). Therefore, the application of variational inequalities to dynamical systems allows us to unify static and dynamic aspects in the study of economic phenomena ([8, 19]). Since both variational inequalities and dynamical systems have been generalized by means of point to set maps, in this paper we focus on the relations among variational inequalities with set-valued operator and differential inclusions. As the study in the single-valued case has dealt with projected DS, we recall in Section 2 the notion of projected differential inclusion (as in [1]), together with the basic results on variational inequalities. Main results are proven in Section 3, where existence of solutions of Minty type variational inequalities is related to the monotonicity of trajectories of a projected differential inclusion. Finally, in Section 4, we apply the results to a generalized gradient inclusion.

2. PRELIMINARIES

We first recall basic results on differential inclusions and variational inequalities. In order to simplify the notation, we need to make the following standing assumptions, which hold throughout the paper unless otherwise stated:

i) K denotes a convex and closed subset of $\mathbb{R}^n$;
ii) F denotes an upper semi-continuous (u.s.c.) map from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$, with nonempty convex and compact values.

For the sake of completeness, we recall the definition of upper semi-continuity for a set-valued map:

Definition 2.1. A map F from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$ is said to be u.s.c. at $x_0 \in \mathbb{R}^n$, if for every open set N containing $F(x_0)$, there exists a neighbourhood M of x_0 such that $F(M) \subseteq N$. F is said to be u.s.c. when it is so at every $x_0 \in \mathbb{R}^n$.

2.1. Differential Inclusions.

We start by recalling from [1] the following result about projection:

Theorem 2.1. *We can associate to every $x \in \mathbb{R}^n$ a unique element $\pi_K(x) \in K$, satisfying:*

$$\|x - \pi_K(x)\| = \min_{y \in K} \|x - y\|.$$

It is characterized by the following inequality:

$$\langle \pi_K(x) - x, \pi_K(x) - y \rangle \le 0, \ \ \forall y \in K.$$

Furthermore the map $\pi_K(\cdot)$ is non expansive, i.e.:

$$\|\pi_K(x) - \pi_K(y)\| \le \|x - y\|.$$

The map π_K is said to be the projector (of best approximation) onto K. When K is a linear subspace, then π_K is linear (see [1]). We set $\pi_K(0) = m(K)$ (i.e. $m(K)$ denotes the element of K with minimal norm). For our aims, we set also:

$$\pi_K(A) = \bigcup_{x \in A} \pi_K(x).$$

The following notation should be common:

$$C^- = \{v \in \mathbb{R}^n : \langle v, a\rangle \le 0, \forall a \in C\}$$

is the (negative) polar cone of the set $C \subseteq \mathbb{R}^n$, while:

$$T(C,x) = \{v \in \mathbb{R}^n : \exists v_n \to v,\ \alpha_n > 0,\ \alpha_n \to 0,\ x + \alpha_n v_n \in C\}$$

is the Bouligand tangent cone to the set C at $x \in \mathrm{cl}C$ and $N(C,x) = [T(C,x)]^-$ stands for the normal cone to C at $x \in \mathrm{cl}C$.

It is known that $T(C,x)$ and $N(C,x)$ are closed sets and $N(C,x)$ is convex. Furthermore, when we consider a closed convex set $K \subseteq \mathbb{R}^n$, then $T(K,x) = \mathrm{cl}\,\mathrm{cone}\,(K - x)$ ($\mathrm{cone}\,A$ denotes the cone generated by the set A), so that the tangent cone is also convex.

Proposition 2.2 ([1]). *Let A be a compact convex subset of $\mathbb{R}^n$, T be a closed convex cone and $N = T^-$ be its polar cone. Then:*

$$\pi_T(A) \subseteq A - N. \tag{2.1}$$

The elements of minimal norm are equal in the two sets:

$$m(\pi_T(A)) = m(A - N)$$

and satisfy:

$$\sup_{z \in -A} \langle z, m(\pi_T(A))\rangle + \|m(\pi_T(A)\|^2 \le 0.$$

We recall that, given a map $G : K \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$, a differential inclusion is the problem of finding an absolutely continuous function $x(\cdot)$, defined on an interval $[0,T]$, such that:

$$\begin{cases} \forall t \in [0,T], & x(t) \in K, \\ \text{for a.a. } t \in [0,T], & x'(t) \in G(x(t)). \end{cases}$$

The solutions of the previous problem are called also *trajectories* of the differential inclusion. Moreover, any $x(\cdot)$ such that:

$$\begin{cases} \forall t \in [0,T], & x(t) \in K, \\ \text{for a.a. } t \in [0,T], & x'(t) = m(G(x(t))) \end{cases}$$

is called a *slow solution* of the differential inclusion.

We are concerned with the following problem, which is a special case of differential inclusion.

Problem 1. Find an absolutely continuous function $x(\cdot)$ from $[0,T]$ into $\mathbb{R}^n$, satisfying:

$$(DVI(F,K)) \qquad \begin{cases} \forall t \in [0,T], & x(t) \in K, \\ \text{for a.a. } t \in [0,T], & x'(t) \in -F(x(t)) - N(K, x(t)) \end{cases}$$

In [1], the previous problem is referred to as a "differential variational inequality" (for short, DVI) and it is proven to be equivalent to a "projected differential inclusion" (for short, PDI).

Theorem 2.3. *The solutions of Problem 1 are the solutions of:*

$$(PDI(F,K)) \qquad \begin{cases} \forall t \in [0,T], & x(t) \in K, \\ \text{for a.a. } t \in [0,T], & x'(t) \in \pi_{T(K,x(t))}\left(-F(x(t))\right), \end{cases}$$

and conversely.

Remark 2.4. We recall that when F is a single-valued operator, then the corresponding "projected differential equation" and its applications have been studied for instance in [8, 19, 20].

Theorem 2.5 ([1]). *The slow solutions of ($DVI(F,K)$) and ($PDI(F,K)$) coincide.*

Definition 2.2. A point $x^* \in K$ is an equilibrium point for ($DVI(F,K)$), when:

$$0 \in -F(x^*) - N(K, x^*).$$

We recall the following existence result.

Theorem 2.6. a) *If K is compact, then there exists an equilibrium point for ($DVI(F,K)$).*

b) *If $m(F(\cdot))$ is bounded, then, for any $x_0 \in K$ there exists an absolutely continuous function $x(t)$ defined on an interval $[0,T]$, such that:*

$$\begin{cases} x(0) = x_0, \quad x'(t) \in -F(x(t)) - N_K(x(t)) \text{ for a.a. } t \in [0,T], \\ \forall t \in [0,T], \quad x(t) \in K. \end{cases}$$

Finally we recall the notion of monotonicity of a trajectory of ($DVI(F,K)$), as stated in [1], which plays a crucial role for our main results.

Definition 2.3. Let V be a function from K to $\mathbb{R}^+$. A trajectory $x(t)$ of ($DVI(F,K)$) is monotone (with respect to V) when:

$$\forall t \geq s, \quad V(x(t)) - V(x(s)) \leq 0.$$

If the previous inequality holds strictly $\forall t > s$, then we say that $x(t)$ is strictly monotone w.r.t. V.

We are mainly concerned with the case when the previous definition applies w.r.t. the function:

$$\tilde{V}_{x^*}(x) = \frac{\|x - x^*\|^2}{2},$$

where x^* is an equilibrium point of ($DVI(F,K)$).

We need also the following result which relates the monotonicity of trajectories and Liapunov functions.

Theorem 2.7 ([1]). *Let K be a subset of $\mathbb{R}^n$ and let $V : K \to \mathbb{R}^+$ be a differentiable function. Assume that for all $x_0 \in K$, there exists $T > 0$ and a trajectory $x(\cdot)$ defined on $[0,T)$ of the differential inclusion $x'(t) \in F(x(t))$, $x(0) = x_0$, satisfying:*

$$\forall s \geq t, \quad V(x(s)) - V(x(t)) \leq 0.$$

Then V is a Liapunov function for F, that is $\forall x \in K$, $\exists \xi \in F(x)$, such that $\langle V'(x), \xi \rangle \leq 0$.

2.2. **Variational Inequalities.** Although we are mainly concerned with Minty type variational inequalities, in this section we also state the Stampacchia variational inequality and exploit some relations between the two formulations. The Minty lemma, which constitutes the main result for this section, legitimizes the Minty formulation we present for the variational inequality. The notation is classical (see for instance [4, 9, 12]):

Definition 2.4. A point $x^* \in K$ is a solution of a Stampacchia Variational Inequality (for short, SVI) when $\exists \xi^* \in F(x^*)$ such that:

$$(SVI(F,K)) \qquad \langle \xi^*, y - x^* \rangle \geq 0, \qquad \forall y \in K.$$

Definition 2.5. A point $x^* \in K$ is a solution of a Strong Minty Variational Inequality (for short, $SMVI$), when:

$$(SMVI(F,K)) \qquad \langle \xi, y - x^* \rangle \geq 0, \qquad \forall y \in K, \ \forall \xi \in F(y).$$

Definition 2.6. A point $x^* \in K$ is a solution of a Weak Minty Variational Inequality (for short, $WMVI$), when $\forall y \in K$, $\exists \xi \in F(y)$ such that:

$$(WMVI(F,K)) \qquad \langle \xi, y - x^* \rangle \geq 0.$$

Definition 2.7. If in Definition 2.5 (resp. 2.6), strict inequality holds $\forall y \in K$, $y \neq x^*$, then we say that x^* is a "strict" solution of ($SMVI(F,K)$) (resp. of ($WMVI(F,K)$)).

Remark 2.8. When F is single valued, Definitions 2.5 and 2.6 reduce to the classical notion of MVI. (see e.g. [2, 21]).

The classical Minty Lemma (see for instance [17]) relates the Minty Variational Inequalities and Stampacchia Variational Inequalities, when F is a single valued operator. The following result gives an extension to the case in which F is a point-to-set map. We recall first the following definition (see e.g. [12]).

Definition 2.8. F is said to be:

i) monotone, if for all $x, y \in K$, we have:

$$\forall u \in F(x),\ \forall v \in F(y):\ \langle v - u, y - x \rangle \geq 0;$$

ii) pseudo-monotone (resp. strictly pseudo-monotone), if for all $x, y \in K$ (resp. for all $x, y \in K$ with $y \neq x$) the following implication holds:

$$\exists u \in F(x) : \langle u, y - x \rangle \geq 0 \Rightarrow \forall v \in F(y) : \langle v, y - x \rangle \geq 0;$$

$$\big(\exists u \in F(x) : \langle u, y - x \rangle \geq 0 \Rightarrow \forall v \in F(y) : \langle v, y - x \rangle > 0\big)$$

Remark 2.9. The following relations among different classes of monotone maps are classical:

$$\begin{array}{ccc} \text{monotone} \Rightarrow & & \text{pseudomonotone} \\ & & \Uparrow \\ & & \text{strictly pseudomonotone.} \end{array}$$

Lemma 2.10. i) *Any $x^* \in K$, which solves ($WMVI(F,K)$), it is a solution of ($SVI(F,K)$) as well.*

ii) *If F is a pseudo-monotone map, any solution of ($SVI(F,K)$) also solves ($SMVI(F,K)$).*

iii) *If F is a strictly pseudo-monotone map, any solution of ($SVI(F,K)$) is a strict solution of ($SMVI(F,K)$).*

Proof. i) Let z be an arbitrary point in K and consider $y = x^* + t(z - x^*) \in K$, where $t \in (0,1)$. Since x^* solves ($WMVI(F,K)$), we have that $\forall t \in (0,1)$, $\exists \xi = \xi(t) \in F(x^* + t(z - x^*))$, such that:

$$\langle \xi(t), t(z - x^*) \rangle \geq 0,$$

that is:

$$\langle \xi(t), z - x^* \rangle \geq 0.$$

Since F is u.s.c., we get that for any integer $n > 0$, there exists a number $\delta_n > 0$ such that, for $t \in (0, \delta_n]$ the following holds:

$$F\big(x^* + t(z - x^*)\big) \subseteq F(x^*) + \frac{1}{n}B.$$

Hence, for $t \in (0, \delta_n]$, $\xi(t) = f(t) + \gamma(t)$, where $f(t) \in F(x^*)$ and $\gamma(t) \in \frac{1}{n}B$. Without loss of generality we can assume $\delta_n < 1\ \forall n$ and we have:

$$0 \leq \langle \xi(t), z - x^* \rangle = \langle f(t), z - x^* \rangle + \langle \gamma(t), z - x^* \rangle.$$

Furthermore, by the Cauchy-Schwartz inequality, we get:

$$|\langle \gamma(t), z - x^* \rangle| \leq \|\gamma(t)\| \, \|z - x^*\| \leq \frac{1}{n} \|z - x^*\|,$$

so that, choosing in particular, $t = \delta_n$, we obtain:

$$\langle f(\delta_n), z - x^* \rangle \geq -\frac{1}{n} \|z - x^*\|.$$

Recalling that $F(x^*)$ is a compact set, when $n \to +\infty$ we can assume that $f(\delta_n) \to \bar{f} \in F(x^*)$ and we get:

$$\langle \bar{f}, z - x^* \rangle \geq 0. \tag{2.2}$$

By the former construction, we have that $\forall z \in K$, there exists $\bar{f} = \bar{f}(z) \in F(x^*)$ such that (2.2) holds.

Since F is convex and compact-valued, then, from Lemma 1 in [3], we get the result. The proof of ii) and iii) is trivial. □

Remark 2.11.

i) Since every solution of ($SMVI(F, K)$) is also a solution of ($WMVI(F, K)$), then, from the previous theorem we obtain that, if F is pseudo-monotone, the solution sets of ($WMVI(F, K)$), ($SMVI(F, K)$) and ($SVI(F, K)$) coincide.

ii) It is easy to prove that if ($SMVI(F, K)$) admits a strict solution x^*, then, x^* is the unique solution of ($SVI(F, K)$).

iii) It is also seen that $x^* \in K$ is an equilibrium point for ($DVI(F, K)$) if and only if it is a solution of ($SVI(F, K)$).

3. VARIATIONAL INEQUALITIES AND MONOTONICITY OF TRAJECTORIES

Our main results concern the relations between the solutions of Minty variational inequalities and the monotonicity of trajectories of ($DVI(F, K)$), w.r.t. the function $\tilde{V}_{x^*}$.

Theorem 3.1. *If $x^* \in K$ is a solution of ($SMVI(F, K)$), then every trajectory $x(t)$ of ($DVI(F, K)$) is monotone w.r.t. function $\tilde{V}_{x^*}$.*

Proof. We observe that, under the hypotheses of the theorem, x^* is an equilibrium point of ($DVI(F, K)$) (recall Lemma 2.10 and Remark 2.11 point iii)). Since $x(t)$ is differentiable a.e., so is $v(t) = \tilde{V}_{x^*}(x(t))$ and we have (at least a.e.):

$$\begin{aligned} v'(t) &= \langle \tilde{V}'_{x^*}(x(t)), x'(t) \rangle \\ &= \langle x'(t), x(t) - x^* \rangle \\ &= \langle -\xi(x(t)) - n_K(x(t)), x(t) - x^* \rangle, \end{aligned}$$

where $\xi(x(t)) \in F(x(t))$ and $n_K(x(t)) \in N(K, x(t)))$. Hence $v'(t) \leq 0$ for a.a. $t \geq 0$ and hence, for $t_2 > t_1$:

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} v'(\tau) d\tau \leq 0.$$

□

Corollary 3.2. *Let x^* be an equilibrium point of ($DVI(F, K)$) and assume that F is pseudo-monotone. Then every trajectory of ($DVI(F, K)$) is monotone w.r.t. function $\tilde{V}_{x^*}$.*

Proof. It is immediate upon combining Lemma 2.10 and Theorem 3.1. □

The following theorem, somehow reverts the previous implication.

Theorem 3.3. *Let x^* be an equilibrium point of ($DVI(F,K)$). If for any point $x \in K$ there exists a trajectory of ($DVI(F,K)$) starting at x and monotone w.r.t. function $\tilde{V}_{x^*}$, then x^* solves ($WMVI(F,K)$).*

Proof. Let $\bar{x} \in \operatorname{ri} K$ (the relative interior of K) be the initial condition for a trajectory $x(t)$ of ($DVI(F,K)$) and assume that $x(t)$ is monotone w.r.t. $\tilde{V}_{x^*}$. If we denote by L the smallest affine subspace generated by K and set $S = L - \bar{x}$, for $x \in K \cap U$, where U is a suitable neighbourhood of $\bar{x}$, we have $T(K,x) = S$ and $N(K,x) = S^\perp$ (the subspace orthogonal to S). So, if $x(t)$ is a trajectory of ($DVI(F,K)$) that starts at $\bar{x}$, then, for t "small enough" (say $t \in [0,T]$), it remains in $\operatorname{ri} K \cap U$ and satisfies (recall Theorem 2.3):

$$\begin{cases} \text{for all } t \in [0,T], & x(t) \in K; \\ \text{for a.a. } t \in [0,T], & x'(t) \in \pi_S(-F(x(t)). \end{cases}$$

Since S is a subspace, π_S is a linear operator; hence $\pi_S(-F(x(t))$ is compact and convex $\forall t \in [0,T]$ and furthermore $\pi_S(-F(\cdot))$ is u.s.c.

Applying Theorem 2.7 we obtain the existence of a vector $\mu \in \pi_S(-F(\bar{x}))$, such that $\langle \tilde{V}'_{x^*}(\bar{x}), \mu \rangle \leq 0$. Taking into account inclusion (2.1), we have $\mu = -\xi(\bar{x}) - n(\bar{x})$, where $\xi(\bar{x}) \in F(\bar{x})$ and $n(\bar{x}) \in S^\perp$. Hence:

$$\begin{aligned} \langle \tilde{V}'_{x^*}(\bar{x}), \mu \rangle &= \langle -\xi(\bar{x}) - n(\bar{x}), \bar{x} - x^* \rangle \\ &= \langle -\xi(\bar{x}), \bar{x} - x^* \rangle + \langle n(\bar{x}), x^* - \bar{x} \rangle \leq 0, \end{aligned}$$

from which it follows, since $\langle n(\bar{x}), x^* - \bar{x} \rangle = 0$:

$$\langle \xi(\bar{x}), \bar{x} - x^* \rangle \geq 0.$$

Since $\bar{x}$ is arbitrary in $\operatorname{ri} K$, we have:

$$\langle \xi(x), x - x^* \rangle \geq 0, \ \ \forall x \in \operatorname{ri} K.$$

Now, let $\tilde{x} \in \operatorname{cl} K \setminus \operatorname{ri} K$. Since $\operatorname{cl} K = \operatorname{cl} \operatorname{ri} K$, then $\tilde{x} = \lim x_k$, for some sequence $\{x_k\} \in \operatorname{ri} K$ and:

$$\langle \xi(x_k), x_k - x^* \rangle \geq 0, \qquad \forall k.$$

There exists a closed ball $\bar{B}(\tilde{x}, \delta)$, with centre in $\tilde{x}$ and radius δ, such that x_k is contained in the compact set $\bar{B}(\tilde{x}, \delta) \cap K$ and since F is u.s.c., with compact images, the set:

$$\bigcup_{y \in \bar{B}(\tilde{x},\delta) \cap K} F(y)$$

is compact (see Proposition 3, p. 42 in [1]) and we can assume that $\xi(x_k) \to \tilde{\xi} \in \bigcup_{y \in \bar{B}(\tilde{x},\delta) \cap K} F(y)$. From the upper semi-continuity of F, it follows also $\tilde{\xi} \in F(\tilde{x})$ and so:

$$\langle \tilde{\xi}, \tilde{x} - x^* \rangle \geq 0.$$

This completes the proof. □

Theorem 3.1 can be strengthened with the following:

Proposition 3.4. *Let x^* be a strict solution of ($SMVI(F,K)$), then:*

i) *x^* is the unique equilibrium point of ($DVI(F,K)$);*
ii) *every trajectory of ($DVI(F,K)$), starting at a point $x_0 \in K$ and defined on $[0,+\infty)$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.*

Proof. The uniqueness of the equilibrium point follows from Remark 2.11 point i). The strict monotonicity of any trajectory $x(t)$ w.r.t. $\tilde{V}_{x^*}$ follows along the lines of the proof of Theorem 3.1. Now the proof of the convergence is an application of Liapunov function's technique.

Let $x(t) \in K$ be a solution of ($DVI(F,K)$), starting at some point $x_0 \in K$, i.e. with $x(0) = x_0$. Assume, ab absurdo, that $\alpha = \lim_{t\to+\infty} v(t) > 0 = \min_{y\in K} \tilde{V}_{x^*}(\cdot)$, where $v(t) = \tilde{V}_{x^*}(x(t))$. We observe that the limit defining α exists, because of the monotonicity of $v(\cdot)$ and to assume it differs from 0, it is equivalent to say that $x(t) \not\to x^*$. Thus, since $x(t)$ is monotone w.r.t. $\tilde{V}_{x^*}$, we have $\forall t \geq 0$:

$$\alpha \leq v(t) \leq \delta = \frac{\|x_0 - x^*\|^2}{2}.$$

Let

$$L := \left\{ x \in K \, : \, \alpha \leq \frac{\|x - x^*\|^2}{2} \leq \delta \right\},$$

we have that L is a compact set and $x^* \notin L$, while $x(t) \in L$, $\forall t \geq 0$. Since x^* is a strict solution of ($SMVI(F,K)$), we have:

$$\langle \xi, y - x^* \rangle < 0, \quad \forall y \in K,\, y \neq x^*, \quad \forall \xi \in -F(y)$$

and, in particular:

$$\langle \xi, y - x^* \rangle < 0, \quad \forall y \in L, \quad \forall \xi \in -F(y).$$

Now, we observe that there exists a number $m > 0$, such that:

$$\max_{\xi \in -F(y)} \langle \xi, y - x^* \rangle \leq -m, \quad \forall y \in L.$$

In fact, if such a number does not exist, we would obtain the existence of sequences $y_n \in L$ and $\xi_n \in F(y_n)$, such that:

$$\langle \xi_n, y_n - x^* \rangle \geq -\frac{1}{n}.$$

Sending n to $+\infty$, we can assume that $y_n \to \bar{y} \in L$. Furthermore, since F is u.s.c. with compact images, the set:

$$\bigcup_{y\in L} F(y)$$

is compact and we can also assume $\xi_n \to \bar{\xi} \in \bigcup_{y\in L} F(y)$. By the upper semi-continuity of F, it follows also $\bar{\xi} \in F(\bar{y})$ and we get the absurdo:

$$\langle \bar{\xi}, \bar{y} - x^* \rangle \geq 0.$$

We have:

$$v'(t) = \langle x'(t), x(t) - x^* \rangle = \langle a(t) + b(t), x(t) - x^* \rangle,$$

with $a(t) \in -F(x(t))$, $b(t) \in -N(K, x(t))$ and hence:

$$v'(t) = \langle a(t), x(t) - x^* \rangle + \langle -b(t), x^* - x(t) \rangle.$$

Since $x(t) \in L$, for $t \geq 0$, we have $\langle a(t), x(t) - x^* \rangle \leq -m$, while $\langle -b(t), x^* - x(t) \rangle \leq 0$. Therefore $v'(t) \leq -m$, for $t \geq 0$. Now, we obtain, for $T > 0$:

$$v(T) - v(0) = \int_0^T v'(\tau) d\tau \leq -mT.$$

If $T = \frac{v(0)}{m}$, we get $v(T) \leq 0 = \min_{y\in K} V(\cdot)$. But we also have:

$$v(T) \geq \alpha > \min_{y\in K} V(\cdot) = 0.$$

Hence a contradiction follows and we must have $\alpha = 0$, that is $x(t) \to x^*$. □

Corollary 3.5. *Let x^* be an equilibrium point of ($DVI(F,K)$) and assume that F is strictly pseudo-monotone. Then properties i) and ii) of the previous proposition hold.*

Proof. It is immediate on combining Lemma 2.10 and Proposition 3.4. □

Example 3.1. Let $K = \mathbb{R}^2$ and consider the system of autonomous differential equations:

$$x'(t) = -F(x(t)),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a single-valued map defined as:

$$F(x,y) = \begin{bmatrix} -y + x|1 - x^2 - y^2| \\ x + y|1 - x^2 - y^2| \end{bmatrix}.$$

Clearly $(x^*, y^*) = (0,0)$ is an equilibrium point and one has $\langle F(x,y),(x,y)\rangle \geq 0\ \forall (x,y) \in \mathbb{R}^2$, so that $(0,0)$ is a solution of ($SMVI(F,K)$) and hence, according to Theorem 3.1, every solution $x(t)$ of the considered system of differential equations is monotone w.r.t. $\tilde{V}_{x^*}$. Anyway, not all the solutions of the system converge to $(0,0)$. In fact, passing to polar coordinates, the system can be written as:

$$\begin{cases} \rho'(t) = -\rho(t)|1 - \rho^2(t)|, \\ \theta'(t) = -1 \end{cases}$$

and solving the system, one can easily see that the solutions that start at a point (ρ,θ), with $\rho \geq 1$ do not converge to $(0,0)$, while the solutions that start at a point (ρ,θ) with $\rho < 1$ converge to $(0,0)$. This last fact could be checked by observing that for every $c < 1$, $(0,0)$ is a strict solution of ($SMVI(F,K_c)$) where:

$$K_c := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq c\}.$$

4. AN APPLICATION: GENERALIZED GRADIENT INCLUSIONS

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on the open set Ω. Equations of the form:

$$x'(t) = -f'(x(t)), \;\; x(0) = x_0$$

are called "gradient equations" (see for instance [13]). In [1] an extension of the classical gradient equation to the case in which f is a lower semi-continuous convex function is considered, replacing the above gradient equation, with the differential inclusion:

$$x'(t) \in -\partial f(x(t)), \quad x(0) = x_0,$$

where ∂f denotes the subgradient of f.

Here, we consider a locally Lipschitz function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, where Ω is an open set containing the closed convex set K, and the DVI:

$$(DVI(\partial_C f, K)) \qquad \begin{cases} \forall t \in [0,T], & x(t) \in K, \\ \text{for a.a. } t \in [0,T], & x'(t) \in -\partial_C f(x(t)) - N(K, x(t)), \end{cases}$$

where $\partial_C f(x)$ denotes Clarke's generalized gradient of f at x [6], with the aim of studying the behaviour of its trajectories. For the sake of completeness we recall the following definitions.

Definition 4.1. Let f be a locally Lipschitz function from K to $\mathbb{R}$. Clarke's generalized gradient of f at x is the subset of $\mathbb{R}^n$, defined as:

$$\partial_C f(x) = \operatorname{conv}\big\{\lim f'(x_k) : x_k \to x, \; f \text{ is differentiable at } x_k\big\}$$

(here f' denotes the gradient of f and $\operatorname{conv} A$ stands for the convex hull of the set $A \subseteq \mathbb{R}^n$).

Definition 4.2 ([16]). We say that $\partial_C f$ is semistrictly pseudo-monotone on K, when for every $x, y \in K$, with $f(x) \neq f(y)$, we have:

$$\exists u \in \partial_C f(x) : \langle u, y - x \rangle \geq 0 \Rightarrow \forall v \in \partial_C f(y) : \langle v, y - x \rangle > 0.$$

Clearly, if $\partial_C f$ is strictly pseudo-monotone, then it is also semistrictly pseudo-monotone.

Definition 4.3. i) f is said to be pseudo-convex on K when $\forall x, y \in K$, with $f(y) > f(x)$, there exists a positive number $a(x, y)$, depending on x and y and a number $\delta(x, y) \in (0, 1]$, such that:

$$f(\lambda x + (1 - \lambda)y) \leq f(y) - \lambda a(x, y), \ \ \forall \lambda \in (0, \delta(x, y)).$$

ii) f is said to be strictly pseudo-convex if the previous inequality holds whenever $f(y) \geq f(x), \ \ x \neq y$.

Theorem 4.1 ([16]). i) *Assume that $\partial_C f$ is semistrictly pseudo-monotone on an open convex set $A \subseteq \mathbb{R}^n$. Then f is pseudo-convex on A.*

ii) *Assume that $\partial_C f$ is strictly pseudo-monotone on an open convex set A. Then f is strictly pseudo-convex on A.*

Remark 4.2. Strictly pseudo-monotone and semistrictly pseudo-monotone maps are called respectively "strictly quasi-monotone" and "semistrictly quasi-monotone" in [16].

Definition 4.4. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is inf-compact on the closed convex set K, when $\forall c \in \mathbb{R}$, the level sets:

$$lev_{\leq c} f := \left\{ x \in K : f(x) \leq c \right\}$$

are compact.

Remark 4.3. Clearly, if f is inf-compact on K the set $\operatorname{argmin}(f, K)$ of minimizers of f over K is compact. The converse does not hold.

Proposition 4.4. *Let $x(t)$ be a slow solution of ($DVI(\partial_C f, K)$) defined on $[0, T]$. Then, $\forall s_1, \ s_2 \in [0, T]$ with $s_2 \geq s_1$, we have:*

$$f(x(s_2)) - f(x(s_1)) \leq - \int_{s_1}^{s_2} \| m(-\partial_C f(x(s)) - N(K, x(s))) \|^2 ds.$$

Hence the function $g(t) = f(x(t))$ is non-increasing and $\lim_{t \to +\infty} f(x(t))$ exists.

Proof. Since a locally Lipschitz function is differentiable a.e., the function $g(t) = f(x(t))$ is differentiable a.e., with $g'(t) = f'(x(t))x'(t)$ and $x'(t) \in m(-\partial_C f(x(t)) - N(K, x(t)))$ for a.a. t . Recalling (Theorem 2.5) that the slow solutions of ($DVI(\partial_C f, K)$) coincide with the slow solutions of $PDI(\partial_C f, K)$ and that $f'(x(t)) \in \partial_C f(x(t))$ [6], we have from Proposition 2.2:

$$\sup_{z \in \partial_C f(x(t))} \langle z, m(-\partial_C f(x(t)) - N(K, x(t))) \rangle + \| m(-\partial_C f(x(t)) - N(K, x(t))) \|^2 \leq 0$$

and for a.a. t, we get:

$$g'(t) = f'(x(t))x'(t) \leq -\| m(-\partial_C f(x(t)) - N(K, x(t))) \|^2 \leq 0,$$

from which we deduce:

$$f(x(s_2)) - f(x(s_1)) \leq - \int_{s_1}^{s_2} \| m(-\partial_C f(x(s)) - N(K, x(s))) \|^2 ds \leq 0.$$

The second part of the theorem is now an immediate consequence. □

Proposition 4.5. *Suppose that f achieves its minimum over K at some point. Assume that $\partial_C f$ is a semistrictly pseudo-monotone map and that f is inf-compact. Then every slow solution $x(t)$ of $(DVI(\partial_C f, K))$ defined on $[0, +\infty)$, is such that:*

$$\lim_{t \to +\infty} f(x(t)) = \min_{x \in K} f(x).$$

Furthermore, every cluster point of $x(t)$ is a minimum point for f over K.

Proof. Let $x(t)$ be a slow solution starting at $x_0 = x(0)$ and ab absurdo, assume that $\lim_{t \to +\infty} f(x(t)) = \alpha > \min_{x \in K} f(x)$. The set:

$$Z = \{x \in K : \alpha \leq f(x) \leq f(x_0)\}.$$

is compact, since f is inf-compact and $\operatorname{argmin}(f, K) \cap Z = \emptyset$. If we set $A = \{x(t), t \in [0, +\infty)\}$, then we get $\operatorname{cl} A \subseteq Z$ (recall Proposition 4.4), and hence $\operatorname{argmin}(f, K) \cap \operatorname{cl} A = \emptyset$. If $x^* \in \operatorname{argmin}(f, K)$, then it is an equilibrium point of $(DVI(\partial_C f, K))$ (see [6]), that is:

$$0 \in \partial_C f(x^*) + N(K, x^*),$$

and this is equivalent (see point iii) of Remark 2.11) to the fact that x^* solves $(SVI(\partial_C f, K))$, that is, to the existence of vector $v \in \partial_C f(x^*)$ such that:

$$\langle v, x - x^* \rangle \geq 0, \ \ \forall x \in K.$$

It follows also: $\langle v, a - x^* \rangle \geq 0, \ \ \forall a \in \operatorname{cl} A$ and since $\partial_C f$ is semistrictly pseudo-monotone, we have (observe that $f(a) \neq f(x^*) \ \forall a \in \operatorname{cl} A$):

$$\langle w, a - x^* \rangle < 0, \ \ \forall w \in -\partial_C f(a), \ \ \forall a \in \operatorname{cl} A.$$

Observing that $\operatorname{cl} A$ is a compact set, as in the proof of Theorem 3.4, it follows that there exists a positive number m such that:

$$\langle w, a - x^* \rangle < -m, \ \ \forall w \in -\partial_C f(a), \ \ \forall a \in \operatorname{cl} A.$$

Hence, letting $v(t) = \frac{\|x(t) - x^*\|^2}{2}$, as in the proof of Theorem 3.4, we obtain $v'(t) \leq -m$ for a.a. t and hence, for $T > 0$:

$$v(T) - v(0) = \int_0^T v'(\tau) d\tau \leq -mT.$$

For $T = v(0)/m$, we obtain $v(T) \leq 0$, that is $v(T) = 0$ and hence $x(T) = x^*$, but this is absurdo, since the set A does not intersect $\operatorname{argmin}(f, K)$.

Now the last assertion of the theorem is obvious. □

The previous result can be strengthened using the results of Section 3.

Proposition 4.6. *Let f be a function that achieves its minimum over K at some point x^* and assume that x^* is a strict solution of $(SMVI(\partial_C f, K))$. Then every solution defined on $[0, +\infty)$ of $(DVI(\partial_C f, K))$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.*

Proof. It is immediate recalling that if x^* is a minimum point for f over K, then it is an equilibrium point of $(DVI(\partial_C f, K))$ and applying Proposition 3.4. □

Remark 4.7. If x^* is a strict solution of $(SMVI(\partial_C f, K))$, then it can be proved that f is strictly increasing along rays starting at x^*. The proof is similar to that of Proposition 4 in [7].

Corollary 4.8. *Let f be a function that achieves its minimum over K at some point x^*. If $\partial_C f$ is strictly pseudo-monotone, then x^* is the unique minimum point for f over K and every solution of $(DVI(\partial_C f, K))$ defined on $[0, +\infty)$ converges to x^*.*

Proof. Recall that, under the hypotheses, f is strictly pseudo-convex (Theorem 4.1) and hence it follows easily that x^* is the unique minimum point of f over K. The proof is now an immediate consequence of Corollary 3.5. □

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