journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 4, Article 117, 4 pp.

AN EXTENSION OF OZAKI AND NUNOKAWA'S UNIVALENCE CRITERION

HORIANA TUDOR
FACULTY OF MATHEMATICS AND INFORMATICS
"TRANSILVANIA" UNIVERSITY
2200 BRAŞOV, ROMANIA
htudor@unitbv.ro

Received 11 May, 2008; accepted 14 October, 2008
Communicated by N.E. Cho

ABSTRACT. In this paper we obtain a sufficient condition for the analyticity and the univalence of the functions defined by an integral operator. In a particular case we find the well known condition for univalency established by S. Ozaki and M. Nunokawa.

Key words and phrases: Univalent functions, Univalence criteria.

2000 *Mathematics Subject Classification.* 30C55.

1. INTRODUCTION

We denote by $U_r = \{z \in \mathbb{C} : \ |z| < r\}$ a disk of the z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0, \infty)$. Let $\mathcal{A}$ be the class of functions f analytic in U such that $f(0) = 0$, $f'(0) = 1$.

Theorem 1.1 ([1]). *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \tag{1.1}$$

then the function f is univalent in U.

2. PRELIMINARIES

In order to prove our main result we need the theory of Löewner chains; we recall the basic result of this theory, from Pommerenke.

Theorem 2.1 ([2]). *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, $a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r.For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $\operatorname{Re} p(z,t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3. MAIN RESULTS

Theorem 3.1. *Let $f \in \mathcal{A}$ and α be a complex number,* $\operatorname{Re}\alpha > 0$*. If the following inequalities*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 \tag{3.1}$$

and

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right)|z|^{2\alpha} + 2\frac{1-|z|^{2\alpha}}{\alpha}\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) + \frac{(1-|z|^{2\alpha})^2}{\alpha^2 |z|^{2\alpha}}\left[\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) + (1-\alpha)\left(\frac{f(z)}{z} - 1\right)\right]\right| \le 1 \tag{3.2}$$

are true for all $z \in U \setminus \{0\}$, then the function F_α,

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{\frac{1}{\alpha}} \tag{3.3}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us consider the function $g_1(z,t)$ given by

$$g_1(z,t) = 1 - \frac{e^{2\alpha t} - 1}{\alpha}\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right).$$

For all $t \in I$ and $z \in U$ we have $e^{-t}z \in U$ and because $f \in \mathcal{A}$, the function $g_1(z,t)$ is analytic in U and $g_1(0,t) = 1$. Then there is a disk $U_{r_1},\ 0 < r_1 < 1$ in which $g_1(z,t) \neq 0$, for all $t \in I$. For the function

$$g_2(z,t) = \alpha \int_0^{e^{-t}z} u^{\alpha-1} f'(u)du\ ,$$

$g_2(z,t) = z^\alpha \cdot g_3(z,t)$, it can be easily shown that $g_3(z,t)$ is analytic in U_{r_1} and $g_3(0,t) = e^{-\alpha t}$. It follows that the function

$$g_4(z,t) = g_3(z,t) + \frac{(e^{\alpha t} - e^{-\alpha t})\left(\frac{f(e^{-t}z)}{e^{-t}z}\right)^2}{g_1(z,t)}$$

is also analytic in a disk $U_{r_2},\ 0 < r_2 \le r_1$ and $g_4(0,t) = e^{\alpha t}$. Therefore, there is a disk $U_{r_3},\ 0 < r_3 \le r_2$ in which $g_4(z,t) \neq 0$, for all $t \in I$ and we can choose an analytic branch of $[g_4(z,t)]^{1/\alpha}$, denoted by $g(z,t)$. We choose the branch which is equal to e^t at the origin.

From these considerations it follows that the function

$$L(z,t) = z \cdot g(z,t) = e^t z + a_2(t)z^2 + \cdots$$

is analytic in U_{r_3}, for all $t \in I$ and can be written as follows

$$L(z,t) = \left[\alpha \int_0^{e^{-t}z} u^{\alpha-1} f'(u)du + \frac{(e^{2\alpha t} - 1)e^{(2-\alpha)t} z^{\alpha-2} f^2(e^{-t}z)}{1 - \frac{e^{2\alpha t}-1}{\alpha}\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right)}\right]^{\frac{1}{\alpha}}. \tag{3.4}$$

From the analyticity of $L(z,t)$ in U_{r_3}, it follows that there is a number $r_4,\ 0 < r_4 < r_3$, and a constant $K = K(r_4)$ such that

$$|L(z,t)/e^t| < K, \qquad \forall z \in U_{r_4}, \quad t \in I,$$

and then $\{L(z,t)/e^t\}$ is a normal family in U_{r_4}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and $r_5,\ 0 < r_5 < r_4$, there exists a constant $K_1 > 0$ (that depends on T and r_5) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1, \qquad \forall z \in U_{r_5}, \quad t \in [0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_5}. We also have that the function

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Bigg/ \frac{\partial L(z,t)}{\partial t}$$

is analytic in U_r, $0 < r < r_5$, for all $t \in I$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U for all $t \in I$, it is sufficient to show that the function $w(z,t)$ defined in U_r by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}$$

can be continued analytically in U and that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

By simple calculations, we obtain

$$w(z,t) = \left(\frac{e^{-2t}z^2 f'(e^{-t}z)}{f^2(e^{-t}z)} - 1\right)e^{-2\alpha t} + 2\frac{1-e^{-2\alpha t}}{\alpha}\left(\frac{e^{-2t}z^2 f'(e^{-t}z)}{f^2(e^{-t}z)} - 1\right) + \frac{(1-e^{-2\alpha t})^2}{\alpha^2 e^{-2\alpha t}}\left[\left(\frac{e^{-2t}z^2 f'(e^{-t}z)}{f^2(e^{-t}z)} - 1\right) + (1-\alpha)\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right)\right]. \tag{3.5}$$

From (3.1) and (3.2) we deduce that the function $w(z,t)$ is analytic in the unit disk and

$$|w(z,0)| = \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1. \tag{3.6}$$

We observe that $w(0,t) = 0$. Let t be a fixed number, $t > 0,\ z \in U,\ z \neq 0$. Since $|e^{-t}z| \le e^{-t} < 1$ for all $z \in \overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ we conclude that the function $w(z,t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each arbitrary fixed $t > 0$, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|, \tag{3.7}$$

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (3.5) we get

$$w(e^{i\theta},t) = \left(\frac{u^2 f'(u)}{f^2(u)} - 1\right)|u|^{2\alpha} + 2\frac{1-|u|^{2\alpha}}{\alpha}\left(\frac{u^2 f'(u)}{f^2(u)} - 1\right) + \frac{(1-|u|^{2\alpha})^2}{\alpha^2|u|^{2\alpha}}\left[\left(\frac{u^2 f'(u)}{f^2(u)} - 1\right) + (1-\alpha)\left(\frac{f(u)}{u} - 1\right)\right].$$

Since $u \in U$, the inequality (3.2) implies that $|w(e^{i\theta},t)| \le 1$ and from (3.6) and (3.7) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$.

From Theorem 2.1 it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U for each $t \in I$, in particular $L(z,0)$. But $L(z,0) = F_\alpha(z)$. Therefore the function $F_\alpha(z)$ defined by (3.3) is analytic and univalent in U. □

If in Theorem 3.1 we take $\alpha = 1$ we obtain the following corollary which is just Theorem 1.1, namely Ozaki-Nunokawa's univalence criterion.

Corollary 3.2. *Let $f \in \mathcal{A}$. If for all $z \in U$, the inequality (3.1) holds true, then the function f is univalent in U.*

Proof. For $\alpha = 1$ we have $F_1(z) = f(z)$ and the inequality (3.2) becomes

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right)\left[|z|^2 + 2(1-|z|^2) + \frac{(1-|z|^2)^2}{|z|^2}\right]\right| = \left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) \cdot \frac{1}{|z|^2}\right| \leq 1. \tag{3.8}$$

It is easy to check that if the inequality (3.1) is true, then the inequality (3.8) is also true. Indeed, the function g,

$$g(z) = \frac{z^2 f'(z)}{f^2(z)} - 1$$

is analytic in U, $g(z) = b_2 z^2 + b_3 z^3 + \cdots$, which shows that $g(0) = g'(0) = 0$. In view of (1.1) we have $|g(z)| < 1$ and using Schwarz's lemma we get $|g(z)| < |z|^2$. □

Example 3.1. Let n be a natural number, $n \geq 2$, and the function

$$f(z) = \frac{z}{1 - \frac{z^{n+1}}{n}}. \tag{3.9}$$

Then f is univalent in U and $F_{\frac{n+1}{2}}$ is analytic and univalent in U, where

$$F_{\frac{n+1}{2}}(z) = \left[\frac{n+1}{2}\int_0^z u^{\frac{n-1}{2}} f'(u)du\right]^{\frac{2}{n+1}}. \tag{3.10}$$

Proof. We have

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = z^{n+1} \tag{3.11}$$

and

$$\frac{f(z)}{z} - 1 = \frac{z^{n+1}}{n - z^{n+1}}. \tag{3.12}$$

It is clear that condition (3.1) of Theorem 3.1 is satisfied, and the function f is univalent in U.

Taking into account (3.11) and (3.12), condition (3.2) of Theorem 3.1 becomes

$$\begin{aligned}
&\Bigg| |z|^{2(n+1)} + \frac{4}{n+1}|z|^{n+1}(1-|z|^{n+1}) + \frac{4}{(n+1)^2}(1-|z|^{n+1})^2 \\
&\qquad + \frac{2(1-n)}{(n+1)^2}(1-|z|^{n+1})^2 \frac{1}{n-|z|^{n+1}}\Bigg| \\
&\leq \frac{1}{(n+1)^2}\left[(n+1)^2|z|^{2(n+1)} + 4(n+1)(1-|z|^{n+1}) + 6(1-|z|^{n+1})^2\right] \\
&= \frac{1}{(n+1)^2}\left[(n^2-2n+3)|z|^{2(n+1)} + (4n-8)|z|^{n+1} + 6\right] \leq 1,
\end{aligned}$$

because the greatest value of the function

$$g(x) = (n^2 - 2n + 3)x^2 + (4n-8)x + 6,$$

for $x \in [0,1],\ n \geq 2$ is taken for $x = 1$ and is $g(1) = (n+1)^2$. Therefore the function $F_{\frac{n+1}{2}}$ is analytic and univalent in U. □

REFERENCES

[1] S. OZAKI AND M. NUNOKAWA, The Schwartzian derivative and univalent functions, *Proc. Amer. Math. Soc.*, **33**(2) (1972), 392–394.

[2] Ch. POMMERENKE, *Univalent Functions*, Vandenhoech Ruprecht, Göttingen, 1975.