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GENERALIZATIONS OF THE TRIANGLE INEQUALITY

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ABSTRACT. The triangle inequality $\|\mathbf{x}+\mathbf{y}\| \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)$ is well-known and fundamental. Since the 8th General Inequalities meeting in Hungary (September 15-21, 2002), the author has been considering an idea that as triangle inequality, the inequality $\|\mathbf{x}+\mathbf{y}\|^2 \leq 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$ may be more suitable. The triangle inequality $\|\mathbf{x}+\mathbf{y}\|^2 \leq 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$ will be naturally generalized for some natural sum of any two members $\mathbf{f}_j$ of any two Hilbert spaces $\mathcal{H}_j$. We shall introduce a natural sum Hilbert space for two arbitrary Hilbert spaces.

Key words and phrases: Triangle inequality, Hilbert space, Sum of two Hilbert spaces, various operators among Hilbert spaces, Reproducing kernel, Linear mapping, Norm inequality.

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1. A GENERAL CONCEPT

Following [1], we shall introduce a general theory for linear mappings in the framework of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then we shall consider the linear mapping

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex-valued functions on E. In order to investigate the linear mapping (1.1), we form a positive matrix $K(p,q)$ on E defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \text{ on } E \times E. \tag{1.2}$$

Then, we obtain the following:

(I) The range of the linear mapping (1.1) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$.

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(II) In general, we have the inequality

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}.$$

Here, for a member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \text{ on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

(III) In general, we have the inversion formula in (1.1) in the form

$$f \to \mathbf{f}^* \tag{1.3}$$

in (II) by using the reproducing kernel Hilbert space $H_K(E)$. However, this formula is, in general, involved and intricate. Indeed, we need specific arguments. Here, we assume that the inversion formula (1.3) may be, in general, established.

Now we shall consider two systems

$$f_j(p) = (\mathbf{f}_j, \mathbf{h}_j(p))_{\mathcal{H}_j}, \ \mathbf{f}_j \in \mathcal{H}_j \tag{1.4}$$

in the above way by using $\{\mathcal{H}_j, E, \mathbf{h}_j\}_{j=1}^2$. Here, we assume that E is the same set for both systems in order to have the output functions $f_1(p)$ and $f_2(p)$ on the same set $E.$

For example, we can consider the usual operator

$$f_1(p) + f_2(p)$$

in $\mathcal{F}(E)$. Then, we can consider the following problem:

How do we represent the sum $f_1(p) + f_2(p)$ on E in terms of their inputs $\mathbf{f}_1$ and $\mathbf{f}_2$ through one system?

We shall see that by using the theory of reproducing kernels we can give a natural answer to this problem. Following similar ideas, we can consider various operators among Hilbert space (see [2] for the details). In particular, for the product of two Hilbert spaces, the idea gives generalizations of convolutions and the related natural convolution norm inequalities. These norm inequalities gave various generalizations and applications to forward and inverse problems for linear mappings in the framework of Hilbert spaces, see for example, [3, 4, 5]. Furthermore, surprisingly enough, for some very general nonlinear systems, we can consider similar problems. For its importance in general inequalities, we shall refer to natural generalizations of the triangle inequality $\|\mathbf{x}+\mathbf{y}\|^2 \leq 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$ for some natural sum of two arbitrary Hilbert spaces. And then, we may consider this inequality to be more suitable as a triangle inequality than the usual triangle inequality.

2. SUM

By (I), $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$, and we note that for the reproducing kernel Hilbert space $H_{K_1+K_2}(E)$ admitting the reproducing kernel

$$K_1(p,q) + K_2(p,q) \text{ on } E,$$

$H_{K_1+K_2}(E)$ is composed of all functions

$$f(p) = f_1(p) + f_2(p); \ f_j \in H_{K_j}(E) \tag{2.1}$$

and its norm $\|f\|_{H_{K_1+K_2}(E)}$ is given by

$$\|f\|^2_{H_{K_1+K_2}(E)} = \min\left\{\|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}\right\} \tag{2.2}$$

where the minimum is taken over $f_j \in H_{K_j}(E)$ satisfying (2.1) for f. Hence, in general, we have the inequality

$$\|f_1 + f_2\|^2_{H_{K_1+K_2}(E)} \leq \|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}. \tag{2.3}$$

In particular, note that for the same K_1 and K_2, we have, for $K_1 = K_2 = K$

$$\|f_1 + f_2\|^2_{H_K(E)} \leq 2(\|f_1\|^2_{H_K(E)} + \|f_2\|^2_{H_K(E)}).$$

Furthermore, the particular inequality (2.3) may be considered as a natural triangle inequality for the sum of reproducing kernel Hilbert spaces $H_{K_j}(E)$.

For the positive matrix $K_1 + K_2$ on E, we assume the expression in the form

$$K_1(p,q) + K_2(p,q) = (\mathbf{h}_S(q), \mathbf{h}_S(p))_{\mathcal{H}_S} \ \text{ on } E \times E \tag{2.4}$$

with a Hilbert space $\mathcal{H}_S$-valued function on E and further we assume that

$$\{\mathbf{h}_S(p); p \in E\} \text{ is complete in } \mathcal{H}_S. \tag{2.5}$$

Such a representation is, in general, possible ([1, page 36 and see Chapter 1, §5]). Then, we can consider the linear mapping from $\mathcal{H}_S$ onto $H_{K_1+K_2}(E)$

$$f_S(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S}, \ \mathbf{f}_S \in \mathcal{H}_S \tag{2.6}$$

and we obtain the isometric identity

$$\|f_S\|_{H_{K_1+K_2}(E)} = \|\mathbf{f}_S\|_{\mathcal{H}_S}. \tag{2.7}$$

Hence, for such representations (2.4) with (2.5), we obtain the isometric relations among the Hilbert spaces $\mathcal{H}_S$.

Now, for the sum $f_1(p) + f_2(p)$ there exists a uniquely determined $\mathbf{f}_S \in \mathcal{H}_S$ satisfying

$$f_1(p) + f_2(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S} \ \text{ on } E. \tag{2.8}$$

Then, $\mathbf{f}_S$ will be considered as a sum of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these mappings and so, we shall introduce the notation

$$\mathbf{f}_S = \mathbf{f}_1[+]\mathbf{f}_2. \tag{2.9}$$

This sum for the members $\mathbf{f}_1 \in \mathcal{H}_1$ and $\mathbf{f}_2 \in \mathcal{H}_2$ is introduced through the three mappings induced by $\{\mathcal{H}_j, \ E, \ \mathbf{h}_j\}$ $(j = 1,2)$ and $\{\mathcal{H}_S, \ E, \ \mathbf{h}_S\}$.

The operator $\mathbf{f}_1[+]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1} + (\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \longrightarrow \mathbf{f}_1[+]\mathbf{f}_2 \tag{2.10}$$

in the sense (II) from $H_{K_1+K_2}(E)$ onto $\mathcal{H}_S$. Then, from (II) and (2.5) we obtain the beautiful triangle inequality

Theorem 2.1. *We have the triangle inequality*

$$\|\mathbf{f}_1[+]\mathbf{f}_2\|^2_{\mathcal{H}_S} \leq \|\mathbf{f}_1\|^2_{\mathcal{H}_1} + \|\mathbf{f}_2\|^2_{\mathcal{H}_2}. \tag{2.11}$$

If $\{\mathbf{h}_j(p); \ p \in E\}$ are complete in $\mathcal{H}_j$ $(j = 1,2)$, then $\mathcal{H}_j$ and H_{K_j} are isometrical, respectively. By using the isometric mappings induced by Hilbert space valued functions $\mathbf{h}_j$ $(j = 1,2)$ and $\mathbf{h}_S$, we can introduce the sum space of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the form

$$\mathcal{H}_1[+]\mathcal{H}_2 \tag{2.12}$$

through the mappings. Of course, the sum is a Hilbert space. Furthermore, such spaces are determined in the framework of isometric relations.

For example, if for some positive number γ

$$K_1 \ll \gamma^2 K_2 \ \text{ on } E \tag{2.13}$$

that is, if $\gamma^2 K_2 - K_1$ is a positive matrix on E, then we have

$$H_{K_1}(E) \subset H_{K_2}(E) \tag{2.14}$$

and

$$\|f_1\|_{H_{K_2}(E)} \leq \gamma \|f_1\|_{H_{K_1}(E)} \text{ for } f_1 \in H_{K_1}(E) \tag{2.15}$$

([1, page 37]). Hence, in this case, we need not to introduce a new Hilbert space $\mathcal{H}_S$ and the linear mapping (2.6) in Theorem 2.1 and we can use the linear mapping

$$(\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2}, \ \mathbf{f}_2 \in \mathcal{H}_2$$

instead of (2.6) in Theorem 2.1.

3. EXAMPLE

We shall consider two linear transforms

$$f_j(p) = \int_T F_j(t)\overline{h(t,p)}\rho_j(t)dm(t), \ p \in E \tag{3.1}$$

where ρ_j are positive continuous functions on T,

$$\int_T |h(t,p)|^2 \rho_j(t)dm(t) < \infty \text{ on } p \in E \tag{3.2}$$

and

$$\int_T |F_j(t)|^2 \rho_j(t)dm(t) < \infty. \tag{3.3}$$

We assume that $\{h(t,p); p \in E\}$ is complete in the spaces satisfying (3.3). Then, we consider the associated reproducing kernels on E

$$K_j(p,q) = \int_T h(t,p)\overline{h(t,p)}\rho_j(t)dm(t)$$

and, for example we consider the expression

$$K_1(p,q) + K_2(p,q) = \int_T h(t,p)\overline{h(t,p)}(\rho_1(t) + \rho_2(t))dm(t). \tag{3.4}$$

So, we can consider the linear transform

$$f(p) = \int_T F(t)\overline{h(t,p)}(\rho_1(t) + \rho_2(t))dm(t) \tag{3.5}$$

for functions F satisfying

$$\int_T |F(t)|^2 (\rho_1(t) + \rho_2(t))dm(t) < \infty.$$

Hence, through the three transforms (3.1) and (3.5) we have the sum

$$(F_1[+]F_2)(t) = \frac{F_1(t)\rho_1(t) + F_2(t)\rho_2(t)}{\rho_1(t) + \rho_2(t)}. \tag{3.6}$$

4. DISCUSSIONS

- Which is more general the classical one and the second triangle inequality (2.11)? We stated: for two arbitrary Hilbert spaces, we can introduce the natural sum and the second triangle inequality (2.11) is valid in a natural way.
- Which is more beautiful the classical one and the second one (2.11)? The author believes the second one is more beautiful. (Gods love two; and 2 is better than 1/2).
- Which is more widely applicable?
- The author thinks the second triangle inequality (2.11) is superior to the classical one as a triangle inequality. However, for the established long tradition, how will be:
  - for the classical one, the triangle inequality of the first kind and
  - for the second inequality (2.11), the triangle inequality of the second kind?

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