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INEQUALITIES FOR AVERAGES OF CONVEX AND SUPERQUADRATIC FUNCTIONS

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ABSTRACT. We consider the averages $A_n(f) = 1/(n-1)\sum_{r=1}^{n-1} f(r/n)$ and $B_n(f) = 1/(n+1)\sum_{r=0}^{n} f(r/n)$. If f is convex, then $A_n(f)$ increases with n and $B_n(f)$ decreases. For the class of functions called superquadratic, a lower bound is given for the successive differences in these sequences, in the form of a convex combination of functional values, in all cases at least $f(1/3n)$. Generalizations are formulated in which r/n is replaced by a_r/a_n and $1/n$ by $1/c_n$. Inequalities are derived involving the sum $\sum_{r=1}^{n}(2r-1)^p$.

Key words and phrases: Inequality, Averages, Convex, Superquadratic, Monotonic.

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1. INTRODUCTION

For a function f, define

$$A_n(f) = \frac{1}{n-1}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \qquad (n \geq 2) \tag{1.1}$$

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and

$$B_n(f) = \frac{1}{n+1}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) \qquad (n \geq 1), \tag{1.2}$$

the averages of values at equally spaced points in $[0, 1]$, respectively, excluding and including the end points. In [2] it was shown that if f is convex, then $A_n(f)$ increases with n, and $B_n(f)$ decreases. A typical application, found by taking $f(x) = -\log x$, is that $(n!)^{1/n}/(n+1)$ decreases with n (this strengthens the result of [6] that $(n!)^{1/n}/n$ is decreasing). Similar results for averages including one end point can be derived, and have appeared independently in [5] and [4].

In this article, we generalize the theorems of [2] in two ways. First, we present a class of functions for which a non-zero lower bound can be given for the differences $A_{n+1}(f) - A_n(f)$ and $B_{n-1}(f) - B_n(f)$. Recall that a convex function satisfies

$$f(y) - f(x) \geq C(x)(y - x)$$

for all x, y, where $C(x) = f'(x)$ (or, if f is not differentiable at x, any number between the left and right derivatives at x). In [1], the authors introduced the class of *superquadratic* functions, defined as follows. A function f, defined on an interval $I = [0, a]$ or $[0, \infty)$, is “superquadratic" if for each x in I, there exists a real number $C(x)$ such that

$$f(y) - f(x) \geq f(|y - x|) + C(x)(y - x) \tag{SQ}$$

for all $y \in I$. For non-negative functions, this amounts to being “more than convex" in the sense specified. The term is chosen because x^p is superquadratic exactly when $p \geq 2$, and equality holds in the definition when $p = 2$. In Section 2, we shall record some of the elementary facts about superquadratic functions. In particular, they satisfy a refined version of Jensen's inequality for sums of the form $\sum_{r=1}^{n} \lambda_r f(x_r)$, with extra terms inserted.

For superquadratic functions, lower bounds for the differences stated are obtained in the form of convex combinations of certain values of f. By the refined Jensen inequality, they can be rewritten in the form $f(1/3n) + S$, where S is another convex combination. These estimates preserve equality in the case $f(x) = x^2$. By a further application of the inequality, we show that S is not less than $f(a/n)$ (for $B_n(f)$), or $f(a/(n+3))$ (for $A_n(f)$), where $a = \frac{16}{81} = (\frac{2}{3})^4$. This simplifies our estimates to the sum of just two functional values, but no longer preserving equality in the case of x^2.

We then present generalized versions in which $f(r/n)$ is replaced by $f(a_r/a_n)$ and $1/(n \pm 1)$ is replaced by $1/c_{n\pm1}$. Under suitable conditions on the sequences (a_n) and (c_n), we show that the generalized $A_n(f)$ and $B_n(f)$ are still monotonic for monotonic convex or concave functions. These theorems generalize and unify results of the same sort in [4], which take one-end-point averages as their starting point. At the same time, the previous lower-bound estimates for superquadratic functions are generalized to this case.

There is a systematic duality between the results for $A_n(f)$ and $B_n(f)$ at every stage, but enough difference in the detail for it to be necessary to present most of the proofs separately.

We finish with some applications of our results to sums and products involving odd numbers. For example, if $S_n(p) = \sum_{r=1}^{n}(2r-1)^p$, then $S_n(p)/(2n+1)(2n-1)^p$ decreases with n for $p \geq 1$, and $S_n(p)/(n+1)(2n-1)^p$ increases with n when $0 < p \leq 1$. Also, if $Q_n = 1\cdot 3 \cdot \dots \cdot (2n-1)$, then $Q_n^{1/(n-1)}/(2n+1)$ decreases with n.

2. SUPERQUADRATIC FUNCTIONS

The definition (SQ) of “superquadratic" was given in the introduction. We say that f is *subquadratic* if $-f$ is superquadratic.

First, some immediate remarks. For $f(x) = x^2$, equality holds in (SQ), with $C(x) = 2x$. Also, the definition, with $y = x$, forces $f(0) \le 0$, from which it follows that one can always take $C(0)$ to be 0. If f is differentiable and satisfies $f(0) = f'(0) = 0$, then one sees easily that the $C(x)$ appearing in the definition is necessarily $f'(x)$.

The definition allows some quite strange functions. For example, any function satisfying $-2 \le f(x) \le -1$ is superquadratic. However, for present purposes, our real interest is in non-negative superquadratic functions. The following lemma shows what these functions are like.

Lemma 2.1. *Suppose that f is superquadratic and non-negative. Then f is convex and increasing. Also, if $C(x)$ is as in (SQ), then $C(x) \ge 0$.*

Proof. Convexity is shown in [1, Lemma 2.2]. Together with $f(0) = 0$ and $f(x) \ge 0$, this implies that f is increasing. As mentioned already, we can take $C(0) = 0$. For $x > 0$ and $y < x$, we can rewrite (SQ) as

$$C(x) \ge \frac{f(x) - f(y) + f(x-y)}{x-y} \ge 0.$$

□

The next lemma (essentially Lemma 3.2 of [1]) gives a simple sufficient condition. We include a sketch of the proof for completeness.

Lemma 2.2. *If $f(0) = f'(0) = 0$ and f' is convex (resp. concave), then f is superquadratic (resp. subquadratic).*

Proof. First, since f' is convex and $f'(0) = 0$, we have $f'(x) \le [x/(x+y)]f'(x+y)$ for $x, y \ge 0$, and hence $f'(x) + f'(y) \le f'(x+y)$ (that is, f' is superadditive). Now let $y > x \ge 0$. Then

$$f(y) - f(x) - f(y-x) - (y-x)f'(x) = \int_0^{y-x} [f'(t+x) - f'(t) - f'(x)]dt \ge 0.$$

Similarly for the case $x > y \ge 0$. □

Hence x^p is superquadratic for $p \ge 2$ and subquadratic for $1 < p \le 2$. (It is also easily seen that x^p is subquadratic for $0 < p \le 1$, with $C(x) = 0$). Other examples of superquadratic functions are $x^2 \log x$, $\sinh x$ and

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le a, \\ (x-a)^2 & \text{for } x > a. \end{cases}$$

The converse of Lemma 2.2 is not true. In [1], it is shown where superquadratic fits into the “scale of convexity" introduced in [3].

The refined Jensen inequality is as follows. Let μ be a probability measure on a set E. Write simply $\int x$ for $\int_E x d\mu$.

Lemma 2.3. *Let x be non-negative and μ-integrable, and let f be superquadratic. Define the (non-linear) operator T by: $(Tx)(s) = \left|x(s) - \int x\right|$. Then*

$$\int (f \circ x) \ge f\left(\int x\right) + \int [f \circ (Tx)].$$

The opposite inequality holds if f is subquadratic.

Proof. Assume f is superquadratic. Write $\int x = \overline{x}$. Then

$$\begin{aligned}\int (f \circ x) - f(\overline{x}) &= \int [f(x(s)) - f(\overline{x})] ds \\ &\geq \int f\left(|x(s) - \overline{x}|\right) ds + C(\overline{x}) \int (x(s) - \overline{x}) ds \\ &= \int (f \circ Tx).\end{aligned}$$

□

In fact, the converse holds: if the property stated in Lemma 2.3 holds for all two-point measure spaces, then f is superquadratic [1, Theorem 2.3].

Note that T is a sublinear operator. Iteration of Lemma 2.3 immediately gives:

Lemma 2.4. *If $x \geq 0$ and f is superquadratic, then for each $k \geq 2$,*

$$\int (f \circ x) \geq f\left(\int x\right) + f\left(\int Tx\right) + \cdots + f\left(\int T^{k-1}x\right) + \int [f \circ (T^k x)].$$

and hence

$$\int (f \circ x) \geq \sum_{k=0}^{\infty} f\left(\int T^k x\right).$$

In this paper, we will be using the discrete case of Lemma 2.3. It may be helpful to restate this case in the style in which it will appear: *Suppose that f is superquadratic. Let $x_r \geq 0$ ($1 \leq r \leq n$) and let $\overline{x} = \sum_{r=1}^{n} \lambda_r x_r$, where $\lambda_r \geq 0$ and $\sum_{r=1}^{n} \lambda_r = 1$. Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\overline{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \overline{x}|).$$

For $x \in \mathbb{R}^n$, now write $x(r)$ for the rth component, and, as usual, $\|x\|_\infty = \max_{1 \leq r \leq n} |x(r)|$. In this discrete situation, for the T defined above, it is easy to show that $\|T^k x\|_\infty$ converges to zero geometrically.

Lemma 2.5. *Let $\lambda = \min_{1 \leq r \leq n} \lambda_r$ and let $x \geq 0$. Then $\|Tx\|_\infty \leq (1-\lambda)\|x\|_\infty$, hence $\|T^k x\|_\infty \leq (1-\lambda)^k \|x\|_\infty$.*

Proof. Note that $|x(r) - x(s)| \leq \|x\|_\infty$ for all r, s. So, for each r,

$$\begin{aligned}(Tx)(r) &= \left| \sum_{s=1}^{n} \lambda_s [x(r) - x(s)] \right| \\ &\leq \sum_{s \neq r} \lambda_s |x(r) - x(s)| \\ &\leq (1 - \lambda_r) \|x\|_\infty.\end{aligned}$$

□

It now follows easily that the second inequality in Lemma 2.4 reverses for subquadratic functions satisfying a condition $f(t) \leq ct^p$ for some $p > 0$. Hence equality holds for $f(x) = x^2$.

Note. It is not necessarily true that $\int Tx \leq \int x$, and hence $\|\cdot\|_\infty$ cannot be replaced by $\|\cdot\|_1$ in Lemma 2.5. Take $\lambda_r = 1/n$ for each r, and let $x = (1, 0, \ldots, 0)$. Then $Tx = \left(1 - \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, giving $\int Tx = 2(n-1)/n^2$.

3. THE BASIC THEOREMS

Throughout the following, the quantities $A_n(f)$ and $B_n(f)$ continue to be defined by (1.1) and (1.2).

Theorem 3.1. *If f is superquadratic on $[0,1]$, then for $n \geq 2$,*

$$A_{n+1}(f) - A_n(f) \geq \sum_{r=1}^{n-1} \lambda_r f(x_r), \tag{3.1}$$

where

$$\lambda_r = \frac{2r}{n(n-1)}, \qquad x_r = \frac{n-r}{n(n+1)}.$$

Further,

$$A_{n+1}(f) - A_n(f) \geq f\left(\frac{1}{3n}\right) + \sum_{r=1}^{n-1} \lambda_r f(y_r), \tag{3.2}$$

where

$$y_r = \frac{|2n-1-3r|}{3n(n+1)}.$$

The opposite inequalities hold if f is subquadratic.

Proof. Write $\Delta_n = (n-1)[A_{n+1}(f) - A_n(f)]$. Then

$$\begin{aligned}
\Delta_n &= \frac{n-1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n+1}\right) - \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \\
&= \sum_{r=1}^{n}\left(\frac{r-1}{n} + \frac{n-r}{n}\right) f\left(\frac{r}{n+1}\right) - \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \\
&= \sum_{r=0}^{n-1} \frac{r}{n} f\left(\frac{r+1}{n+1}\right) + \sum_{r=1}^{n-1} \frac{n-r}{n} f\left(\frac{r}{n+1}\right) - \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \\
&= \sum_{r=1}^{n-1} \frac{r}{n}\left[f\left(\frac{r+1}{n+1}\right) - f\left(\frac{r}{n}\right)\right] + \sum_{r=1}^{n-1} \frac{n-r}{n}\left[f\left(\frac{r}{n+1}\right) - f\left(\frac{r}{n}\right)\right].
\end{aligned}$$

We apply the definition of superquadratic to both the differences appearing in the last line, noting that

$$\frac{r+1}{n+1} - \frac{r}{n} = \frac{n-r}{n(n+1)}.$$

We obtain

$$\Delta_n \geq \sum_{r=1}^{n-1} \frac{r}{n} f\left(\frac{n-r}{n(n+1)}\right) + \sum_{r=1}^{n-1} \frac{n-r}{n} f\left(\frac{r}{n(n+1)}\right) + \sum_{r=1}^{n-1} h_r C\left(\frac{r}{n}\right),$$

where

$$h_r = \frac{r}{n}\cdot\frac{r+1}{n+1} + \frac{n-r}{n}\cdot\frac{r}{n+1} - \frac{r}{n} = 0,$$

hence

$$\Delta_n \geq 2\sum_{r=1}^{n-1} \frac{r}{n} f\left(\frac{n-r}{n(n+1)}\right),$$

which is equivalent to (3.1).

We now apply Lemma 2.3. Note that

$$\sum_{r=1}^{n-1} r(n-r) = \frac{1}{2}(n-1)n^2 - \frac{1}{6}(n-1)n(2n-1)$$
$$= \frac{1}{6}(n-1)n(n+1),$$

hence $\sum_{r=1}^{n-1} \lambda_r x_r = 1/3n$ (denote this by $\overline{x}$). So

$$x_r - \overline{x} = \frac{n-r}{n(n+1)} - \frac{1}{3n} = \frac{2n-3r-1}{3n(n+1)},$$

and inequality (3.2) follows. □

The proof of the dual result for $B_n(f)$ follows similar lines, but since the algebraic details are critical, we set them out in full.

Theorem 3.2. *If f is superquadratic on $[0,1]$, then for $n \geq 2$,*

$$B_{n-1}(f) - B_n(f) \geq \sum_{r=1}^{n} \lambda_r f(x_r), \tag{3.3}$$

where

$$\lambda_r = \frac{2r}{n(n+1)}, \qquad x_r = \frac{n-r}{n(n-1)}.$$

Further,

$$B_{n-1}(f) - B_n(f) \geq f\left(\frac{1}{3n}\right) + \sum_{r=1}^{n} \lambda_r f(y_r), \tag{3.4}$$

where

$$y_r = \frac{|2n+1-3r|}{3n(n-1)}.$$

The opposite inequalities hold if f is subquadratic.

Proof. Let $\Delta_n = (n+1)[B_{n-1}(f) - B_n(f)]$. Then

$$\Delta_n = \frac{n+1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n-1}\right) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right)$$
$$= \sum_{r=0}^{n-1} \left(\frac{r+1}{n} + \frac{n-r}{n}\right) f\left(\frac{r}{n-1}\right) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right)$$
$$= \sum_{r=1}^{n} \frac{r}{n} f\left(\frac{r-1}{n-1}\right) + \sum_{r=0}^{n-1} \frac{n-r}{n} f\left(\frac{r}{n-1}\right) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right)$$
$$= \sum_{r=1}^{n} \frac{r}{n}\left[f\left(\frac{r-1}{n-1}\right) - f\left(\frac{r}{n}\right)\right] + \sum_{r=0}^{n-1} \frac{n-r}{n}\left[f\left(\frac{r}{n-1}\right) - f\left(\frac{r}{n}\right)\right].$$

Apply the definition of superquadratic, noting that

$$\left|\frac{r-1}{n-1} - \frac{r}{n}\right| = \frac{n-r}{n(n-1)}.$$

We obtain

$$\Delta_n \geq \sum_{r=1}^{n} \frac{r}{n} f\left(\frac{n-r}{n(n-1)}\right) + \sum_{r=0}^{n-1} \frac{n-r}{n} f\left(\frac{r}{n(n-1)}\right) + \sum_{r=0}^{n} k_r C\left(\frac{r}{n}\right),$$

where

$$k_r = \frac{r}{n} \cdot \frac{r-1}{n-1} + \frac{n-r}{n} \cdot \frac{r}{n-1} - \frac{r}{n} = 0,$$

hence

$$\Delta_n \geq 2 \sum_{r=1}^{n} \frac{r}{n} f\left(\frac{n-r}{n(n-1)}\right),$$

which is equivalent to (3.3). Exactly as in Theorem 3.1, we see that $\sum_{r=1}^{n} \lambda_r x_r = 1/3n$, and (3.4) follows. □

Remark 3.3. These proofs, simplified by not introducing the functional values of f on the right-hand side, reproduce Theorems 1 and 2 of [2] for convex functions.

Remark 3.4. Since these inequalities reverse for subquadratic functions, they become equalities for $f(x) = x^2$, which is both superquadratic and subquadratic. In this sense, they are optimal for the hypotheses: nothing has been lost. However, this is at the cost of fairly complicated expressions. Clearly, if f is also non-negative, then we have the simple lower estimate $f(1/3n)$. in both results. In the case $f(x) = x^2$, it is easily seen that

$$A_n(f) = \frac{1}{3} - \frac{1}{6n}, \qquad B_n(f) = \frac{1}{3} + \frac{1}{6n},$$

hence

$$A_{n+1}(f) - A_n(f) = \frac{1}{6n(n+1)}, \qquad B_{n-1}(f) - B_n(f) = \frac{1}{6n(n-1)},$$

so the term $f(1/3n) = 1/9n^2$ gives about two thirds of the true value.

Averages including one end-point. Let

$$D_n(f) = \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right), \qquad E_n(f) = \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right).$$

If $f(0) = 0$, then

$$D_n(f) = \frac{n-1}{n} A_n(f), \qquad E_n(f) = \frac{n+1}{n} B_n(f).$$

For an increasing, convex function f, we can add a constant to ensure that $f(0) = 0$, and it follows that $D_n(f)$ is increasing and $E_n(f)$ is decreasing ([2, Theorem 3A]; also, with direct proof, [5] and [4]). Further, we have

$$D_{n+1}(f) - D_n(f) = \frac{n}{n+1}[A_{n+1}(f) - A_n(f)] + \frac{1}{n(n+1)} A_n(f)$$

and

$$E_{n-1}(f) - E_n(f) = \frac{n}{n-1}[B_{n-1}(f) - B_n(f)] + \frac{1}{n(n-1)} B_n(f).$$

For non-negative, superquadratic f, we automatically have $f(0) = 0$, so we can read off lower bounds for these differences from the corresponding ones for $A_n(f)$ and $B_n(f)$. With regard to the second term, note that for convex functions, we always have $A_n(f) \geq A_2(f) = f(\frac{1}{2})$ and $B_n(f) \geq \int_0^1 f$.

4. ESTIMATES IN TERMS OF TWO FUNCTIONAL VALUES

For non-negative superquadratic functions, we now give lower estimates for the second term in (3.2) and (3.4) in the form of the value at one point, at the cost of losing exactness for the function $f(x) = x^2$. We shall prove:

Theorem 4.1. *If f is superquadratic and non-negative, then for $n \geq 3$,*

$$A_{n+1}(f) - A_n(f) \geq f\left(\frac{1}{3n}\right) + f\left(\frac{16}{81(n+3)}\right).$$

Theorem 4.2. *If f is superquadratic and non-negative, then for all $n \geq 2$,*

$$B_{n-1}(f) - B_n(f) \geq f\left(\frac{1}{3n}\right) + f\left(\frac{16}{81n}\right).$$

The factor $\frac{16}{81}$ seems a little less strange if regarded as $(\frac{2}{3})^4$.

We give the proof for $B_n(f)$ first, since there are some extra complications in the case of $A_n(f)$. Let λ_r and y_r be as in Theorem 3.2. By Lemma 2.3, discarding the extra terms arising from the definition of superquadratic, we have $\sum_{r=1}^n \lambda_r f(y_r) \geq f(y)$, where $y = y(n) = \sum_{r=1}^n \lambda_r y_r$. We give a lower bound for $y(n)$.

Lemma 4.3. *Let $S = \sum_{r=1}^n r|2n+1-3r|$. Let m be the greatest integer such that $3m \leq 2n+1$. Then*

$$S = 2m(m+1)(n-m).$$

Proof. For *any* m,

$$\begin{aligned}\sum_{r=1}^m r(2n+1-3r) &= \frac{1}{2}m(m+1)(2n+1) - \frac{1}{2}m(m+1)(2m+1)\\ &= m(m+1)(n-m).\end{aligned}$$

In particular, $\sum_{r=1}^n r(2n+1-3r) = 0$. With m now as stated, it follows that

$$\begin{aligned}S &= \sum_{r=1}^m r(2n+1-3r) + \sum_{r=m+1}^n r(3r-2n-1)\\ &= 2\sum_{r=1}^m r(2n+1-3r)\\ &= 2m(m+1)(n-m).\end{aligned}$$

□

Conclusion of the proof of Theorem 4.2. With this notation, we have

$$y(n) = \frac{2S}{3n^2(n+1)(n-1)}.$$

If we insert $3m \leq 2n+1$ and $n - m \leq \frac{1}{3}(n+1)$, we obtain $y(n) \geq (2 - \frac{1}{n})(8/81n)$, not quite the stated result. However, $3m$ is actually one of $2n-1,\ 2n,\ 2n+1$. The exact expressions for $y(n)$ in the three cases, are, respectively:

$$\frac{8}{81n}\cdot\frac{(2n-1)(n+1)}{(n-1)n}, \qquad \frac{8}{81}\cdot\frac{2n+3}{n^2-1}, \qquad \frac{8}{81n}\cdot\frac{(2n+1)(n+2))}{n(n+1)}.$$

In each case, it is clear that $y(n) \geq 16/81n$. □

We now return to Theorem 4.1. Let λ_r and y_r be as defined in Theorem 3.1.

Lemma 4.4. *Let* $S = \sum_{r=1}^{n-1} r|2n-1-3r|$*, and let* m *be the smallest integer such that* $3m \geq 2n-1$*. Then*

$$S = 2(m-1)m(n-m).$$

Proof. Similar to Lemma 4.3, using the fact that (for any m):

$$\sum_{r=1}^{m-1} r(2n-1-3r) = (m-1)m(n-m).$$

□

Conclusion of the proof of Theorem 4.1.
Case $3m = 2n-1$ (so that $n = 2, 5, \ldots$). Then

$$y(n) = \frac{8}{81} \cdot \frac{(n-2)(2n-1)}{n^2(n-1)}.$$

The statement $y(n) \geq 16/[81(n+3)]$ is equivalent to $3n^2 - 13n + 6 \geq 0$, which occurs for all $n \geq 4$.
Case $3m = 2n$ (so $n = 3, 6, \ldots$). Then

$$y(n) = \frac{8}{81} \cdot \frac{(2n-3)}{(n+1)(n-1)},$$

which is not less than $16/[81(n+3)]$ when $3n \geq 7$.
Case $3m = 2n+1$ (so $n = 4, 7, \ldots$). Then

$$y(n) = \frac{8}{81} \frac{(n-1)(2n+1)}{n^2(n+1)}.$$

This time we note that $y(n) \geq 16/[81(n+2)]$ is equivalent to $n^2 - 3n - 2 \geq 0$, which occurs for all $n \geq 4$. □

Note. More precisely, the proof shows that $y(2) = 0$, $y(3) = \frac{1}{27}$ and $y(5) = \frac{2}{75}$, while in all other cases $y(n) \geq 16/[81(n+2)]$.

In principle, the process can be iterated, as in Lemma 2.4. After complicated evaluations, one finds that the next term is of the order of $f(1/30n)$.

5. GENERALIZED VERSIONS

We now formulate generalized versions of the earlier results in which $f(r/n)$ is replaced by $f(a_r/a_n)$ and $1/(n \pm 1)$ is replaced by $1/c_{n\pm1}$, under suitable conditions on the sequences (a_n) and (c_n). For increasing convex functions, we show that the generalized $A_n(f)$ and $B_n(f)$ are still monotonic. There are companion results for decreasing or concave functions, with some of the hypotheses reversed. The results of [4] follow as special cases. For superquadratic functions, we obtain suitable generalizations of the lower bounds given in (3.1) and (3.3).

Theorem 5.1.

(i) *Let* $(a_n)_{n\geq1}$ *and* $(c_n)_{n\geq0}$ *be sequences such that* $a_n > 0$ *and* $c_n > 0$ *for* $n \geq 1$ *and:*
  (A1) $c_0 = 0$ *and* c_n *is increasing,*
  (A2) $c_{n+1} - c_n$ *is decreasing for* $n \geq 0$,
  (A3) $c_n(a_{n+1}/a_n - 1)$ *is decreasing for* $n \geq 1$.

Given a function f, let

$$A_n[f,(a_n),(c_n)] = A_n(f) = \frac{1}{c_{n-1}} \sum_{r=1}^{n-1} f\left(\frac{a_r}{a_n}\right)$$

for $n \geq 2$. Suppose that f is convex, non-negative, increasing and differentiable on an interval J including all the points a_r/a_n for $r < n$. Then $A_n(f)$ increases with n.

(ii) *Suppose that f is decreasing on J and that (A3) is reversed, with the other hypotheses unchanged. Then $A_n(f)$ increases with n.*

(iii) *Suppose that f is concave, non-negative and increasing on J, and that (A2) and (A3) are both reversed, with the other hypotheses unchanged. Then $A_n(f)$ decreases with n.*

Proof. First, consider case (i). Let

$$\Delta_n = c_{n-1}[A_{n+1}(f) - A_n(f)] = \frac{c_{n-1}}{c_n} \sum_{r=1}^{n} f\left(\frac{a_r}{a_{n+1}}\right) - \sum_{r=1}^{n-1} f\left(\frac{a_r}{a_n}\right).$$

We follow the proof of Theorem 3.1, with appropriate substitutions. At the first step, where we previously expressed $n-1$ as $(r-1)+(n-r)$, we now use (A2): we have $c_r - c_{r-1} \geq c_n - c_{n-1}$, hence

$$c_{n-1} \geq c_{r-1} + (c_n - c_r)$$

for $r < n$. Using only the fact that f is non-negative, the previous steps then lead to

$$\Delta_n \geq \sum_{r=1}^{n-1} \frac{c_r}{c_n}\left[f\left(\frac{a_{r+1}}{a_{n+1}}\right) - f\left(\frac{a_r}{a_n}\right)\right] + \sum_{r=1}^{n-1} \frac{c_n - c_r}{c_n}\left[f\left(\frac{a_r}{a_{n+1}}\right) - f\left(\frac{a_r}{a_n}\right)\right]. \tag{5.1}$$

(The condition $c_0 = 0$ is needed at the last step).

For $x, y \in J$, we have $f(y) - f(x) \geq C(x)(y-x)$, where $C(x) = f'(x) \geq 0$. So

$$\Delta_n \geq \sum_{r=1}^{n-1} h_r C\left(\frac{a_r}{a_n}\right),$$

where, by (A3),

$$\begin{aligned} h_r &= \frac{c_r}{c_n} \cdot \frac{a_{r+1}}{a_{n+1}} + \frac{c_n - c_r}{c_n} \cdot \frac{a_r}{a_{n+1}} - \frac{a_r}{a_n} \\ &= \frac{a_r}{c_n a_{n+1}}\left(c_r \frac{a_{r+1}}{a_r} + c_n - c_r - c_n \frac{a_{n+1}}{a_n}\right) \\ &\geq 0. \end{aligned}$$

In case (ii), we have $C(x) \leq 0$, and by reversing (A3), we ensure that $h_r \leq 0$.

In case (iii), the reversal of (A2) has the effect of reversing the inequality in (5.1). We now have $f(y) - f(x) \leq C(x)(y-x)$, with $C(x) \geq 0$, and the reversal of (A3) again gives $h_r \leq 0$. □

The theorem simplifies pleasantly when $c_n = a_n$, because condition (A3) now says the same as (A2).

Corollary 5.2. *Let $(a_n)_{n\geq 0}$ be an increasing sequence with $a_0 = 0$ and $a_1 > 0$. Let f be increasing and non-negative on J. Let $A_n(f)$ be as above, with $c_n = a_n$. If $a_{n+1} - a_n$ is decreasing and f is convex, then $A_n(f)$ increases with n. If $a_{n+1} - a_n$ is increasing and f is concave, then $A_n(f)$ decreases with n.*

We note that the term c_0 does not appear in the definition of $A_n(f)$. Its role is only to ensure that $c_2 - c_1 \le c_1$. Also, the differentiability condition is only to avoid infinite gradient at any point a_r/a_n that coincides with an end point of J.

Simply inserting the definition of superquadratic, we obtain:

Theorem 5.3. *Let (a_n), (c_n) and $A_n(f)$ be as in Theorem 5.1(i). Suppose that f is superquadratic and non-negative on J. Then*

$$A_{n+1}(f) - A_n(f) \ge \frac{1}{c_n c_{n-1}} \sum_{r=1}^{n-1} c_r f\left(\left|\frac{a_{r+1}}{a_{n+1}} - \frac{a_r}{a_n}\right|\right) + \frac{1}{c_n c_{n-1}} \sum_{r=1}^{n-1} (c_n - c_r) f\left(\left|\frac{a_r}{a_n} - \frac{a_r}{a_{n+1}}\right|\right).$$

Note that if (a_n) is increasing, then there is clearly no need for the second modulus sign in Theorem 5.3. Furthermore, it is easily checked that, with the other hypotheses, this implies that a_{n+1}/a_n is decreasing, so that the first modulus sign is redundant as well.

We now formulate the dual results for $B_n(f)$. We need an extra hypothesis, (B4).

Theorem 5.4.

(i) *Let $(a_n)_{n\ge 0}$ and $(c_n)_{n\ge 0}$ be sequences such that $a_n > 0$ and $c_n > 0$ for $n \ge 1$ and:*
  (B1) *$c_0 = 0$ and c_n is increasing,*
  (B2) *$c_n - c_{n-1}$ is increasing for $n \ge 1$,*
  (B3) *$c_n(1 - a_{n-1}/a_n)$ is increasing for $n \ge 1$,*
  (B4) *either $a_0 = 0$ or (a_n) is increasing.*
  Given a function f, let

$$B_n[f, (a_n), (c_n)] = B_n(f) = \frac{1}{c_{n+1}} \sum_{r=0}^{n} f\left(\frac{a_r}{a_n}\right).$$

for $n \ge 1$. Suppose that f is convex, non-negative, increasing and differentiable on an interval J including all the points a_r/a_n for $1 \le r \le n$. Then $B_n(f)$ decreases with n.

(ii) *Suppose that f is decreasing on J and that (B3) and (B4) are both reversed, with the other hypotheses unchanged. Then $B_n(f)$ decreases with n.*

(iii) *Suppose that f is concave, non-negative and increasing on J, and that (B2), (B3), (B4) are all reversed, with the other hypotheses unchanged. Then $B_n(f)$ increases with n.*

Proof. We adapt the proof of Theorem 3.2. For $n \ge 2$, let

$$\Delta_n = c_{n+1}[B_{n-1}(f) - B_n(f)] = \frac{c_{n+1}}{c_n} \sum_{r=0}^{n-1} f\left(\frac{a_r}{a_{n-1}}\right) - \sum_{r=0}^{n} f\left(\frac{a_r}{a_n}\right).$$

Using (B2) in the form $c_{n+1} \ge c_{r+1} + (c_n - c_r)$, together with the non-negativity of f, we obtain

$$\Delta_n \ge \sum_{r=1}^{n-1} \frac{c_r}{c_n}\left[f\left(\frac{a_{r-1}}{a_{n-1}}\right) - f\left(\frac{a_r}{a_n}\right)\right] + \sum_{r=0}^{n-1} \frac{c_n - c_r}{c_n}\left[f\left(\frac{a_r}{a_{n-1}}\right) - f\left(\frac{a_r}{a_n}\right)\right]. \tag{5.2}$$

Separating out the term $r = 0$, we now have in case (i)

$$\Delta_n \ge \sum_{r=1}^{n-1} k_r C\left(\frac{a_r}{a_n}\right) + \delta_n,$$

where $\delta_n = f(a_0/a_{n-1}) - f(a_0/a_n)$. Condition (B4) ensures that $\delta_n \geq 0$ (note that we do not need differentiability at the point a_0/a_n), and (B3) gives

$$\begin{aligned} k_r &= \frac{c_r}{c_n} \cdot \frac{a_{r-1}}{a_{n-1}} + \frac{c_n - c_r}{c_n} \cdot \frac{a_r}{a_{n-1}} - \frac{a_r}{a_n} \\ &= \frac{a_r}{c_n a_{n-1}} \left(c_r \frac{a_{r-1}}{a_r} + c_n - c_r - c_n \frac{a_{n-1}}{a_n} \right) \\ &\geq 0. \end{aligned}$$

In case (ii), the reversed hypotheses give $C(x) \leq 0$, $k_r \leq 0$ and $\delta_n \geq 0$.

In case (iii), the inequality in (5.2) is reversed, and $C(x) \geq 0$, $k_r \leq 0$ and $\delta_n \leq 0$. □

Corollary 5.5. *Let $(a_n)_{n\geq 0}$ be an increasing sequence with $a_0 = 0$ and $a_1 > 0$. Let f be increasing and non-negative on J. Let $B_n(f)$ be as above, with $c_n = a_n$. If $a_n - a_{n-1}$ is increasing and f is convex, then $B_n(f)$ decreases with n. If $a_n - a_{n-1}$ is decreasing and f is concave, then $B_n(f)$ increases with n.*

Theorem 5.6. *Let (a_n), (c_n) and $B_n(f)$ be as in Theorem 5.4(i). Suppose that f is superquadratic and non-negative on J. Then*

$$B_{n-1}(f) - B_n(f) \geq \frac{1}{c_n c_{n+1}} \sum_{r=1}^{n-1} c_r f\left(\left| \frac{a_r}{a_n} - \frac{a_{r-1}}{a_{n-1}} \right| \right) + \frac{1}{c_n c_{n+1}} \sum_{r=0}^{n-1} (c_n - c_r) f\left(\left| \frac{a_r}{a_{n-1}} - \frac{a_r}{a_n} \right| \right).$$

Relation to the theorems of [4]. The theorems of [4] (in some cases, slightly strengthened) are cases of our Theorems 5.1 and 5.4. More exactly, by taking $c_n = n$ in Theorem 5.1, we obtain Theorem 2 of [4], strengthened by replacing $1/n$ by $1/(n-1)$. By taking $c_n = a_{n+1}$ in Theorem 5.1, we obtain Theorem 3 of [4]; of course, the hypothesis fails to simplify as in Corollary 5.2. Theorems A and B of [4] bear a similar relationship to our Theorem 5.4. In the way seen in Section 3, results for one-end-point averages (or their generalized forms) can usually be derived from those for $A_n(f)$ and $B_n(f)$. Also, one-end-point averages lead to more complication in the proofs: ultimately, this can be traced to the fact that the analogues of the original h_r and k_r no longer cancel to zero. All these facts indicate that $A_n(f)$ and $B_n(f)$ are the natural averages for this study.

At this level of generality, it is hardly worth formulating generalizations of the original (3.2) and (3.4) for superquadratic functions. However, in some particular cases one can easily calculate the term corresponding to the previous $f(1/3n)$. For example, in Theorem 5.3, with $c_n = n$ and $a_n = 2n - 1$, we obtain the lower estimate $f(x_n)$, where

$$x_n = \frac{4n+1}{3(4n^2-1)}.$$

Remark 5.7. Our proofs extend without change to three sequences: let

$$A_n(f) = \frac{1}{c_{n-1}} \sum_{r=1}^{n-1} f\left(\frac{a_r}{b_n}\right), \qquad B_n(f) = \frac{1}{c_{n+1}} \sum_{r=0}^{n} f\left(\frac{a_r}{b_n}\right).$$

Conditions (A3) and (B3) become, respectively,

$$c_r(a_{r+1}/a_r - 1) \geq c_n(b_{n+1}/b_n - 1) \quad \text{for } r < n,$$
$$c_n(1 - b_{n-1}/b_n) \geq c_r(1 - a_{r-1}/a_r) \quad \text{for } r \leq n.$$

Condition (B4) becomes: either $a_0 = 0$ or (b_n) is increasing.

6. APPLICATIONS TO SUMS AND PRODUCTS INVOLVING ODD NUMBERS

Let

$$S_n(p) = \sum_{r=1}^{n} (2r-1)^p.$$

Note that $S_n(1) = n^2$. We write also $S_n^*(p) = S_n(p) - 1$. It is shown in [2, Proposition 12] that $S_n(p)/n^{p+1}$ increases with n if $p \geq 1$ or $p < 0$, and decreases with n if $0 \leq p \leq 1$. (This result is derived from a theorem on mid-point averages $\frac{1}{n}\sum_{r=1}^{n} f[(2r-1)/2n]$ requiring both f and its derivative to be convex or concave; note however that it is trivial for $p \leq -1$.) We shall apply our theorems to derive some companion results for $S_n(p)$ and $S_n^*(p)$.

Note first that if $c_n = n$ and $a_n = 2n+1$, then

$$c_n\left(\frac{a_{n+1}}{a_n} - 1\right) = c_n\left(1 - \frac{a_{n-1}}{a_n}\right) = \frac{2n}{2n+1},$$

which increases with n. If $c_n = n$ and $a_n = 2n-1$, then

$$c_n\left(\frac{a_{n+1}}{a_n} - 1\right) = c_n\left(1 - \frac{a_{n-1}}{a_n}\right) = \frac{2n}{2n-1},$$

which decreases with n.

Proposition 6.1. *If $p \geq 1$, then*

$$\frac{S_n(p)}{(2n+1)(2n-1)^p} \quad \textit{decreases with n},$$

$$\frac{S_n^*(p)}{(2n-1)(2n+1)^p} \quad \textit{increases with n}.$$

Proof. Let $f(x)$ be the convex function x^p. The first statement is given by Corollary 5.5, with $a_0 = 0$ and $a_n = 2n-1$ for $n \geq 1$. The second one is given by Corollary 5.2, with $a_0 = 0$ and $a_n = 2n+1$ for $n \geq 1$. □

The case $p = 1$ shows that we cannot replace $S_n^*(p)$ by $S_n(p)$ in the second statement. Also, this statement does not follow in any easy way from the theorem of [2].

The sense in which reversal occurs at $p = 1$ is seen in the next result. Also, we can formulate two companion statements (corresponding ones were not included in Proposition 6.1, because they would be weaker than the given statements).

Proposition 6.2. *If $0 < p \leq 1$, then*

$$\frac{S_n(p)}{(2n-1)(2n+1)^p} \quad \textit{and} \quad \frac{S_n^*(p)}{(n-1)(2n+1)^p} \quad \textit{decrease with n},$$

and

$$\frac{S_n^*(p)}{(2n+1)(2n-1)^p} \quad \textit{and} \quad \frac{S_n(p)}{(n+1)(2n-1)^p} \quad \textit{increase with n}.$$

Proof. The function $f(x) = x^p$ is now concave. The first decreasing expression is given by Corollary 5.2 with $a_0 = 0$ and $a_n = 2n-1$ for $n \geq 1$. The second one is given by Theorem 5.1(iii) with $c_n = n$ and $a_n = 2n+1$.

The first increasing expression is given by Corollary 5.5 with $a_0 = 0$ and $a_n = 2n+1$ for $n \geq 1$. The second one is given by Theorem 5.4(iii) with $c_n = n$ and $a_0 = 0$, $a_n = 2n-1$ for $n \geq 1$. Recall that differentiability at 0 is not required. □

Proposition 6.3. *If $p > 0$, then*

$$\frac{(2n+1)^p}{n-1} S_n^*(-p)$$

increases with n.

Proof. Apply Theorem 5.1(ii) to the decreasing convex function $f(x) = x^{-p}$, with $c_n = n$ and $a_n = 2n + 1$. □

We remark that, unlike [2, Proposition 12], this statement is not trivial when $p = 1$. Again, we cannot replace $S_n^*(p)$ by $S_n(p)$.

Finally, we derive a result for the product $Q_n = 1 \cdot 3 \cdot \cdots \cdot (2n-1)$. It follows from [2, Theorem 4] that $Q_n^{1/n}/n$ decreases with n (though this is not stated explicitly in [2]). Our variant is less neat to state than the theorem of [2], but not a consequence of it.

Proposition 6.4. *The quantity $\frac{1}{2n+1} Q_n^{1/(n-1)}$ decreases with n.*

Proof. Take $f(x) = -\log x$, which is decreasing, convex and non-negative on $(0, 1)$. Again apply Theorem 5.1(ii) with $c_n = n$ and $a_n = 2n + 1$. (Alternatively, we can apply Theorem 5.1(iii) to $f(x) = \log x + K$, where K is chosen so that $\log(1/2n) + K > 0$.) □

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