

# $L_p$ INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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*Abstract:* Let  $D_\alpha P(z)$  denote the polar derivative of a polynomial  $P(z)$  of degree  $n$  with respect to real or complex number  $\alpha$ . If  $P(z)$  does not vanish in  $|z| < k, k \geq 1$ , then it has been proved that for  $|\alpha| \geq 1$  and  $p > 0$ ,

$$\|D_\alpha P\|_p \leq \left( \frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p.$$

An analogous result for the class of polynomials having no zero in  $|z| > k, k \leq 1$  is also obtained.

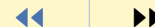
$L_p$  Inequalities for the  
Polar Derivative

Nisar A. Rather

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[Title Page](#)

[Contents](#)



Page 1 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

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# Contents

1	Introduction and Statement of Results	3
2	Lemmas	10
3	Proofs of the Theorems	14



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*L<sub>p</sub>* Inequalities for the  
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vol. 9, iss. 4, art. 103, 2008

---

Title Page

Contents



Page 2 of 21

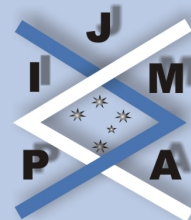
Go Back

Full Screen

Close

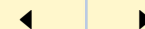
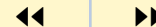
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Title Page

Contents



Page 3 of 21

Go Back

Full Screen

Close

## 1. Introduction and Statement of Results

Let  $P_n(z)$  denote the space of all complex polynomials  $P(z)$  of degree  $n$ . For  $P \in P_n$ , define

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$\|P\|_\infty := \max_{|z|=1} |P(z)|.$$

If  $P \in P_n$ , then

$$(1.1) \quad \|P'\|_\infty \leq n \|P\|_\infty$$

and

$$(1.2) \quad \|P'\|_p \leq n \|P\|_p.$$

Inequality (1.1) is a well-known result of S. Bernstein (see [12] or [15]), whereas inequality (1.2) is due to Zygmund [16]. Arestov [1] proved that the inequality (1.2) remains true for  $0 < p < 1$  as well. Equality in (1.1) and (1.2) holds for  $P(z) = az^n$ ,  $a \neq 0$ . If we let  $p \rightarrow \infty$  in (1.2), we get inequality (1.1).

If we restrict ourselves to the class of polynomials  $P \in P_n$  having no zero in  $|z| < 1$ , then both the inequalities (1.1) and (1.2) can be improved. In fact, if  $P \in P_n$  and  $P(z) \neq 0$  for  $|z| < 1$ , then (1.1) and (1.2) can be, respectively, replaced by

$$(1.3) \quad \|P'\|_\infty \leq \frac{n}{2} \|P\|_\infty$$

and

$$(1.4) \quad \|P'\|_p \leq \frac{n}{\|1+z\|_p} \|P\|_p, \quad p \geq 1.$$



Title Page

Contents



Page 4 of 21

Go Back

Full Screen

Close

Inequality (1.3) was conjectured by P. Erdős and later verified by P. D. Lax [10] whereas the inequality (1.4) was discovered by De Bruijn [5]. Rahman and Schmeisser [13] proved that the inequality (1.4) remains true for  $0 < p < 1$  as well. Both the estimates are sharp and equality in (1.3) and (1.4) holds for  $P(z) = az^n + b$ ,  $|a| = |b|$ .

Malik [11] generalized inequality (1.3) by proving that if  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$ , then

$$(1.5) \quad \|P'\|_\infty \leq \frac{n}{1+k} \|P\|_\infty.$$

Govil and Rahman [8] extended inequality (1.5) to the  $L_p$ -norm by proving that if  $P \in P_n$  and  $P(z) \neq 0$  for  $|z| < k$  where  $k \geq 1$ , then

$$(1.6) \quad \|P'\|_p \leq \frac{n}{\|k+z\|_p} \|P\|_p, \quad p \geq 1.$$

It was shown by Gardner and Weems [7] and independently by Rather [14] that the inequality (1.6) remains true for  $0 < p < 1$  as well.

Let  $D_\alpha P(z)$  denote the polar derivative of polynomial  $P(z)$  of degree  $n$  with respect to a real or complex number  $\alpha$ . Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Polynomial  $D_\alpha P(z)$  is of degree at most  $n - 1$ . Furthermore, the polar derivative  $D_\alpha P(z)$  generalizes the ordinary derivative  $P'(z)$  in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to  $z$  for  $|z| \leq R$ ,  $R > 0$ .



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 21

Go Back

Full Screen

Close

A. Aziz [2] extended inequalities (1.1) and (1.3) to the polar derivative of a polynomial and proved that if  $P \in P_n$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$ ,

$$(1.7) \quad \|D_\alpha P\|_\infty \leq n |\alpha| \|P\|_\infty$$

and if  $P \in P_n$  and  $P(z) \neq 0$  for  $|z| < 1$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$ ,

$$(1.8) \quad \|D_\alpha P\|_\infty \leq \frac{n}{2} (|\alpha| + 1) \|P\|_\infty.$$

Both the inequalities (1.7) and (1.8) are sharp. If we divide both sides of (1.7) and (1.8) by  $|\alpha|$  and let  $|\alpha| \rightarrow \infty$ , we get inequalities (1.1) and (1.3) respectively.

A. Aziz [2] also considered the class of polynomials  $P \in P_n$  having no zero in  $|z| < k$  and proved that if  $P \in P_n$  and  $P(z) \neq 0$  for  $|z| < k$  where  $k \geq 1$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$ ,

$$(1.9) \quad \|D_\alpha P\|_\infty \leq n \left( \frac{|\alpha| + k}{1 + k} \right) \|P\|_\infty.$$

The result is best possible and equality in (1.9) holds for  $P(z) = (z + k)^n$  where  $\alpha$  is any real number with  $\alpha \geq 1$ .

It is natural to seek an  $L_p$  - norm analog of the inequality (1.7). In view of the  $L_p$  - norm extension (1.2) of inequality (1.1), one would expect that if  $P \in P_n$ , then

$$(1.10) \quad \|D_\alpha P\|_p \leq n |\alpha| \|P\|_p,$$

is the  $L_p$  - norm extension of (1.7) analogous to (1.2). Unfortunately, inequality (1.10) is not, in general, true for every complex number  $\alpha$ . To see this, we take in



Title Page

Contents



Page 6 of 21

Go Back

Full Screen

Close

particular  $p = 2$ ,  $P(z) = (1 - iz)^n$  and  $\alpha = i\delta$  where  $\delta$  is any positive real number such that

$$(1.11) \quad 1 \leq \delta < \frac{n + \sqrt{2n(2n-1)}}{3n-2},$$

then from (1.10), by using Parseval's identity, we get, after simplification

$$n(1 + \delta)^2 \leq 2(2n - 1)\delta^2.$$

This inequality can be written as

$$(1.12) \quad \left( \delta - \frac{n + \sqrt{2n(2n-1)}}{3n-2} \right) \left( \delta - \frac{n - \sqrt{2n(2n-1)}}{3n-2} \right) \geq 0.$$

Since  $\delta \geq 1$ , we have

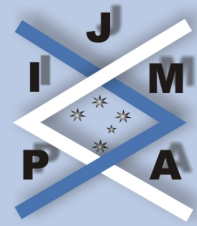
$$\begin{aligned} \left( \delta - \frac{n - \sqrt{2n(2n-1)}}{3n-2} \right) &\geq \left( 1 - \frac{n - \sqrt{2n(2n-1)}}{3n-2} \right) \\ &= \left( \frac{2(n-1) + \sqrt{2n(2n-1)}}{3n-2} \right) > 0 \end{aligned}$$

and hence from (1.12), it follows that

$$\left( \delta - \frac{n + \sqrt{2n(2n-1)}}{3n-2} \right) \geq 0.$$

This gives

$$\delta \geq \frac{n + \sqrt{2n(2n-1)}}{3n-2},$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 7 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

which clearly contradicts (1.11). Hence inequality (1.10) is not, in general, true for all polynomials of degree  $n \geq 1$ .

While seeking the desired extension of inequality (1.8) to the  $L_p$ -norm, recently Govil et al. [9] have made an incomplete attempt by claiming to have proved that if  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < 1$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$ , and  $p \geq 1$ ,

$$(1.13) \quad \|D_\alpha P\|_p \leq n \left( \frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p.$$

A. Aziz, N.A. Rather and Q. Aliya [4] pointed out an error in the proof of inequality (1.13) given by Govil et al. [9] and proved a more general result which not only validated inequality (1.13) but also extended inequality (1.6) for the polar derivative of a polynomial  $P \in P_n$ . In fact, they proved that if  $P \in P_n$  and  $P(z) \neq 0$  for  $|z| < k$  where  $k \geq 1$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$  and  $p \geq 1$ ,

$$(1.14) \quad \|D_\alpha P\|_p \leq n \left( \frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p.$$

The main aim of this paper is to obtain certain  $L_p$  inequalities for the polar derivative of a polynomial valid for  $0 < p < \infty$ . We begin by proving the following extension of inequality (1.2) to the polar derivatives.

**Theorem 1.1.** *If  $P \in P_n$ , then for every complex number  $\alpha$  and  $p > 0$ ,*

$$(1.15) \quad \|D_\alpha P\|_p \leq n(|\alpha| + 1) \|P\|_p.$$

*Remark 1.* If we divide the two sides of (1.15) by  $|\alpha|$  and make  $|\alpha| \rightarrow \infty$ , we get inequality (1.2) for each  $p > 0$ .



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 8 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

As an extension of inequality (1.6) to the polar derivative of a polynomial, we next present the following result which includes inequalities (1.13) and (1.14) for each  $p > 0$  as a special cases.

**Theorem 1.2.** *If  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$  and  $p > 0$ ,*

$$(1.16) \quad \|D_\alpha P\|_p \leq n \left( \frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p.$$

In the limiting case, when  $p \rightarrow \infty$ , the above inequality is sharp and equality in (1.16) holds for  $P(z) = (z + k)^n$  where  $\alpha$  is any real number with  $\alpha \geq 1$ .

The following result immediately follows from Theorem 1.2 by taking  $k = 1$ .

**Corollary 1.3.** *If  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < 1$ , then for every complex number  $\alpha$  with  $|\alpha| \geq 1$  and  $p > 0$ ,*

$$(1.17) \quad \|D_\alpha P\|_p \leq n \left( \frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p.$$

*Remark 2.* Corollary 1.3 not only validates inequality (1.13) for  $p \geq 1$  but also extends it for  $0 < p < 1$  as well.

*Remark 3.* If we let  $p \rightarrow \infty$  in (1.16), we get inequality (1.9). Moreover, inequality (1.6) also follows from Theorem 1.2 by dividing the two sides of inequality (1.16) by  $|\alpha|$  and then letting  $|\alpha| \rightarrow \infty$ .

We also prove:





Title Page

Contents



Page 9 of 21

Go Back

Full Screen

Close

**Theorem 1.4.** If  $P \in P_n$  and  $P(z)$  has all its zeros in  $|z| \leq k$  where  $k \leq 1$  and  $P(0) \neq 0$ , then for every complex number  $\alpha$  with  $|\alpha| \leq 1$  and  $p > 0$ ,

$$(1.18) \quad \|D_\alpha P\|_p \leq n \left( \frac{|\alpha| + k}{\|k + z\|_p} \right) \|P\|_p.$$

In the limiting case, when  $p \rightarrow \infty$ , the above inequality is sharp and equality in (1.18) holds for  $P(z) = (z + k)^n$  for any real  $\alpha$  with  $0 \leq \alpha \leq 1$ .

The following result is an immediate consequence of Theorem 1.4.

**Corollary 1.5.** If  $P \in P_n$  and  $P(z)$  has all its zeros in  $|z| \leq k$  where  $k \leq 1$ , then for every complex number  $\alpha$  with  $|\alpha| \leq 1$ ,

$$\|D_\alpha P\|_\infty \leq n \left( \frac{|\alpha| + k}{1 + k} \right) \|P\|_\infty.$$

The result is best possible and equality in (1.18) holds for  $P(z) = (z + k)^n$  for any real  $\alpha$  with  $0 \leq \alpha \leq 1$ .

Finally, we prove the following result.

**Theorem 1.6.** If  $P \in P_n$  is self-inversive, then for every complex number  $\alpha$  and  $p > 0$ ,

$$\|D_\alpha P\|_p \leq n \left( \frac{|\alpha| + 1}{\|1 + z\|_p} \right) \|P\|_p.$$

The above inequality extends a result due to Dewan and Govil [6] for the polar derivatives.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 21

Go Back

Full Screen

Close

## 2. Lemmas

For the proof of these theorems, we need the following lemmas.

**Lemma 2.1 ([2]).** *If  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$ , then for every real or complex number  $\gamma$  with  $|\gamma| \geq 1$ ,*

$$|D_{\gamma k} P(z)| \leq k |D_{\gamma/k} Q(z)| \quad \text{for } |z| = 1$$

where  $Q(z) = z^n \overline{P(1/\bar{z})}$ .

Setting  $\alpha = \gamma k$  where  $k \geq 1$  in Lemma 2.1, we immediately get:

**Lemma 2.2.** *If  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$ , then for every real or complex number  $\alpha$  with  $|\alpha| \geq 1$ ,*

$$|D_{\alpha} P(z)| \leq k |D_{\alpha/k^2} Q(z)| \quad \text{for } |z| = 1$$

where  $Q(z) = z^n \overline{P(1/\bar{z})}$ .

**Lemma 2.3.** *If  $P \in P_n$  and  $P(z) \neq 0$  in  $|z| < k$  where  $k \geq 1$  and  $Q(z) = z^n \overline{P(1/\bar{z})}$ , then for  $|z| = 1$ ,*

$$k |P'(z)| \leq |Q'(z)|.$$

Lemma 2.3 is due to Malik [9].

**Lemma 2.4.** *If  $P \in P_n$  and  $P(z) \neq 0$  in  $|z| < k$  where  $k \geq 1$  and  $Q(z) = z^n \overline{P(1/\bar{z})}$ , then for every real  $\beta$ ,  $0 \leq \beta < 2\pi$ ,*

$$|k^2 P'(z) + e^{i\beta} Q'(z)| \leq k |P'(z) + e^{i\beta} Q'(z)| \quad \text{for } |z| = 1.$$



Title Page

Contents



Page 11 of 21

Go Back

Full Screen

Close

*Proof of Lemma 2.4.* By hypothesis,  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$  and  $Q(z) = z^n \overline{P(1/\bar{z})}$ . Therefore, by Lemma 2.3, we have

$$k^2 |P'(z)|^2 \leq |Q'(z)|^2 \quad \text{for } |z| = 1.$$

Multiplying both sides of this inequality by  $(k^2 - 1)$  and rearranging the terms, we get

$$(2.1) \quad k^4 |P'(z)|^2 + |Q'(z)|^2 \leq k^2 |P'(z)|^2 + k^2 |Q'(z)|^2 \quad \text{for } |z| = 1.$$

Adding  $2 \operatorname{Re} \left( k^2 P'(z) \overline{Q'(z) e^{i\beta}} \right)$  to the both sides of (2.1), we obtain for  $|z| = 1$ ,

$$|k^2 P'(z) + e^{i\beta} Q'(z)|^2 \leq k^2 |P'(z) + e^{i\beta} Q'(z)|^2 \quad \text{for } |z| = 1$$

and hence

$$|k^2 P'(z) + e^{i\beta} Q'(z)| \leq k |P'(z) + e^{i\beta} Q'(z)| \quad \text{for } |z| = 1.$$

This proves Lemma 2.4.  $\square$

**Lemma 2.5.** If  $P \in P_n$  and  $Q(z) = z^n \overline{P(1/\bar{z})}$ , then for every  $p > 0$  and  $\beta$  real,  $0 \leq \beta < 2\pi$ ,

$$\int_0^{2\pi} \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})|^p d\theta d\beta \leq 2\pi n^p \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta.$$

Lemma 2.5 is due to the author [14] (see also [3]).

**Lemma 2.6.** If  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$  and  $Q(z) = z^n \overline{P(1/\bar{z})}$ , then for every complex number  $\alpha$ ,  $\beta$  real,  $0 \leq \beta < 2\pi$ , and  $p > 0$ ,

$$\int_0^{2\pi} \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})|^p d\theta d\beta \leq 2\pi n^p (|\alpha| + k)^p \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 21

Go Back

Full Screen

Close

*Proof of Lemma 2.6.* We have  $Q(z) = z^n \overline{P(1/\bar{z})}$ , therefore,  $P(z) = z^n \overline{Q(1/\bar{z})}$  and it can be easily verified that for  $0 \leq \theta < 2\pi$ ,

$$nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) = e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} \quad \text{and} \quad nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) = e^{i(n-1)\theta} \overline{P'(e^{i\theta})}.$$

Also, since  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k, k \geq 1$ , therefore,  $Q \in P_n$ . Hence for every complex number  $\alpha, \beta$  real,  $0 \leq \beta < 2\pi$ , we have

$$\begin{aligned} & |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})| \\ &= \left| (nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + k^2 e^{i\beta} (nQ(e^{i\theta}) + \left(\frac{\alpha}{k^2} - e^{i\theta}\right) Q'(e^{i\theta}))) \right| \\ &= \left| (nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta})) + k^2 e^{i\beta} (nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta})) \right. \\ &\quad \left. + \alpha (P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})) \right| \\ &= \left| \left( e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} + k^2 e^{i\beta} e^{i(n-1)\theta} \overline{P'(e^{i\theta})} \right) + \alpha (P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})) \right| \\ &\leq |\alpha| |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})| + |k^2 P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})|. \end{aligned}$$

This gives, with the help of Lemma 2.4,

$$\begin{aligned} |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})| &\leq |\alpha| |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})| \\ &\quad + k |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})| \\ &= (|\alpha| + k) |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})|, \end{aligned}$$

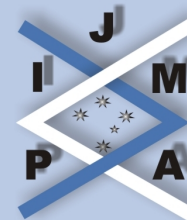
which implies for each  $p > 0$ ,

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})|^p d\theta d\beta \\ & \leq (|\alpha| + k)^p \int_0^{2\pi} \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\beta} Q'(e^{i\theta})|^p d\theta d\beta. \end{aligned}$$

Combining this with Lemma 2.5, we get

$$\int_0^{2\pi} \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})|^p d\theta d\beta \\ \leq 2\pi n^p (|\alpha| + k)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

This completes the proof of Lemma 2.6.  $\square$



*L<sub>p</sub>* Inequalities for the  
Polar Derivative

Nisar A. Rather

vol. 9, iss. 4, art. 103, 2008

Title Page

Contents



Page 13 of 21

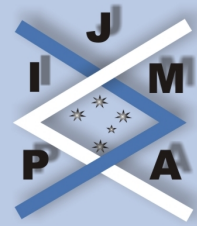
Go Back

Full Screen

Close

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mathematics

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Title Page

Contents



Page 14 of 21

Go Back

Full Screen

Close

### 3. Proofs of the Theorems

*Proof of Theorem 1.1.* Let  $Q(z) = z^n \overline{P(1/\bar{z})}$ , then  $P(z) = z^n \overline{Q(1/\bar{z})}$  and (as before) for  $0 \leq \theta < 2\pi$ , we have

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} \quad \text{and} \quad nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta} \overline{P'(e^{i\theta})},$$

which implies for every complex number  $\alpha$  and  $\beta$  real,  $0 \leq \beta < 2\pi$ ,

$$\begin{aligned} & |D_\alpha P(e^{i\theta}) + e^{i\beta} \{nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\}| \\ &= |nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + e^{i\beta} \{nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) + \alpha Q'(e^{i\theta})\}| \\ &= | \{nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\} + e^{i\beta} \{nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta})\} \\ &\quad + \alpha \{P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\} | \\ &= |e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} + e^{i\beta} e^{i(n-1)\theta} \overline{P'(e^{i\theta})} + \alpha \{P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\}| \\ &\leq |e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} + e^{i\beta} e^{i(n-1)\theta} \overline{P'(e^{i\theta})}| + |\alpha| |P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})| \\ &= (|\alpha| + 1) |P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})|. \end{aligned}$$

This gives with the help of Lemma 2.5 for each  $p > 0$ ,

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + e^{i\beta} \{nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\}|^p d\theta d\beta \\ & \leq (|\alpha| + 1)^p \int_0^{2\pi} \int_0^{2\pi} |P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})|^p d\theta d\beta \\ (3.1) \quad & \leq 2\pi n^p (|\alpha| + 1)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 15 of 21

Go Back

Full Screen

Close

Now using the fact that for any  $p > 0$ ,

$$\int_0^{2\pi} |a + be^{i\beta}|^p d\beta \geq 2\pi \max(|a|^p, |b|^p),$$

(see [5, Inequality (2.1)]), it follows from (3.1) that

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n(|\alpha| + 1) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}, \quad p > 0.$$

This completes the proof of Theorem 1.1. □

*Proof of Theorem 1.2.* Since  $P \in P_n$  and  $P(z)$  does not vanish in  $|z| < k$  where  $k \geq 1$ , by Lemma 2.2, we have for every real or complex number  $\alpha$  with  $|\alpha| \geq 1$ ,

$$(3.2) \quad |D_\alpha P(z)| \leq k |D_{\alpha/k^2} Q(z)| \quad \text{for } |z| = 1,$$

where  $Q(z) = z^n \overline{P(1/\bar{z})}$ . Also, by Lemma 2.6, for every real or complex number  $\alpha$ ,  $p > 0$  and  $\beta$  real,

$$(3.3) \quad \int_0^{2\pi} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})|^p d\beta \right\} d\theta \\ \leq 2\pi n^p (|\alpha| + k)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

Now for every real  $\beta$ ,  $0 \leq \beta < 2\pi$  and  $R \geq r \geq 1$ , we have

$$|R + e^{i\beta}| \geq |r + e^{i\beta}|,$$

which implies

$$\int_0^{2\pi} |R + e^{i\beta}|^p d\beta \geq \int_0^{2\pi} |r + e^{i\beta}|^p d\beta, \quad p > 0.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 16 of 21

Go Back

Full Screen

Close

If  $D_\alpha P(e^{i\theta}) \neq 0$ , we take  $R = k^2 |D_{\alpha/k^2} Q(e^{i\theta})| / |D_\alpha P(e^{i\theta})|$  and  $r = k$ , then by (3.2),  $R \geq r \geq 1$ , and we get

$$\begin{aligned} & \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta})|^p d\beta \\ &= |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} \left| \frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} e^{i\beta} + 1 \right|^p d\beta \\ &= |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} \left| \left| \frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} \right| e^{i\beta} + 1 \right|^p d\beta \\ &= |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} \left| \left| \frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} \right| + e^{i\beta} \right|^p d\beta \\ &\geq |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} |k + e^{i\beta}|^p d\beta. \end{aligned}$$

For  $D_\alpha P(e^{i\theta}) = 0$ , this inequality is trivially true. Using this in (3.3), we conclude that for every real or complex number  $\alpha$  with  $|\alpha| \geq 1$  and  $p > 0$ ,

$$\int_0^{2\pi} |k + e^{i\beta}|^p d\beta \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \leq 2\pi n^p (|\alpha| + k)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta,$$

which immediately leads to (1.16) and this completes the proof of Theorem 1.2.  $\square$

*Proof of Theorem 1.4.* By hypothesis, all the zeros of polynomial  $P(z)$  of degree  $n$  lie in  $|z| \leq k$  where  $k \leq 1$  and  $P(0) \neq 0$ . Therefore, if  $Q(z) = z^n \overline{P(1/\bar{z})}$ , then  $Q(z)$  is a polynomial of degree  $n$  which does not vanish in  $|z| < (1/k)$ , where  $(1/k) \geq 1$ . Applying Theorem 1.2 to the polynomial  $Q(z)$ , we get for every real or





Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 17 of 21

Go Back

Full Screen

Close

complex number  $\beta$  with  $|\beta| \geq 1$  and  $p > 0$ ,

$$(3.4) \quad \left\{ \int_0^{2\pi} |D_\beta Q(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left( \frac{|\beta| + \frac{1}{k}}{\|z + \frac{1}{k}\|_p} \right) \left\{ \int_0^{2\pi} |Q(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}.$$

Now since

$$|Q(e^{i\theta})| = |P(e^{i\theta})|, \quad 0 \leq \theta < 2\pi$$

and

$$\left\| z + \frac{1}{k} \right\|_p = \frac{1}{k} \|z + k\|_p,$$

it follows that (3.4) is equivalent to

$$(3.5) \quad \left\{ \int_0^{2\pi} |D_\beta Q(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left( \frac{k|\beta| + 1}{\|z + k\|_p} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}.$$

Also, we have for every  $\beta$  with  $|\beta| \geq 1$  and  $0 \leq \theta < 2\pi$ ,

$$\begin{aligned} |D_\beta Q(e^{i\theta})| &= |nQ(e^{i\theta}) + (\beta - e^{i\theta})Q'(e^{i\theta})| \\ &= \left| ne^{in\theta} \overline{P(e^{i\theta})} + (\beta - e^{i\theta}) \left( ne^{i(n-1)\theta} \overline{P(e^{i\theta})} - e^{i(n-2)\theta} \overline{P'(e^{i\theta})} \right) \right| \\ &= \left| \beta \left( n\overline{P(e^{i\theta})} - e^{i\theta} \overline{P'(e^{i\theta})} \right) + \overline{P'(e^{i\theta})} \right| \\ &= \left| \overline{\beta} \left( nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) \right) + P'(e^{i\theta}) \right| \\ &= \left| \overline{\beta} \right| \left| D_{1/\overline{\beta}} P(e^{i\theta}) \right|. \end{aligned}$$



Title Page

Contents



Page 18 of 21

Go Back

Full Screen

Close

Using this in (3.5), we get for  $|\beta| \geq 1$ ,

$$(3.6) \quad \left\{ \int_0^{2\pi} |\beta| \left| D_{1/\bar{\beta}} P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \leq n \left( \frac{k|\beta| + 1}{\|z + k\|_p} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}, \quad p > 0.$$

Replacing  $1/\bar{\beta}$  by  $\alpha$  so that  $|\alpha| \leq 1$ , we obtain from (3.6)

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left( \frac{|\alpha| + k}{\|z + k\|_p} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}},$$

for  $|\alpha| \leq 1$  and  $p > 0$ . This proves Theorem 1.4. □

*Proof of Theorem 1.6.* Since  $P(z)$  is a self inversive polynomial of degree  $n$ ,  $P(z) = Q(z)$  for all  $z \in \mathbb{C}$  where  $Q(z) = z^n \overline{P(1/\bar{z})}$ . This gives for every complex number  $\alpha$ ,

$$|D_\alpha P(z)| = |D_\alpha Q(z)|, \quad z \in \mathbb{C}$$

so that

$$(3.7) \quad \left| D_\alpha Q(e^{i\theta}) / D_\alpha P(e^{i\theta}) \right| = 1, \quad 0 \leq \theta < 2\pi.$$

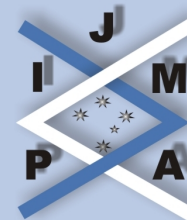
Also, since  $Q(z)$  is a polynomial of degree  $n$ , then

$$(3.8) \quad D_\alpha Q(e^{i\theta}) = nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) + \alpha Q'(e^{i\theta}).$$

Combining (3.1) and (3.8), it follows that for every complex number  $\alpha$  and  $p > 0$ ,

$$(3.9) \quad \int_0^{2\pi} \int_0^{2\pi} |D_\alpha P(e^{i\theta}) + D_\alpha Q(e^{i\theta})|^p d\theta d\beta \\ \leq 2\pi n^p (|\alpha| + 1)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

Using (3.7) in (3.9) and proceeding similarly as in the proof of Theorem 1.2, we immediately get the conclusion of Theorem 1.6.  $\square$



*L<sub>p</sub>* Inequalities for the  
Polar Derivative

Nisar A. Rather

vol. 9, iss. 4, art. 103, 2008

Title Page

Contents



Page 19 of 21

Go Back

Full Screen

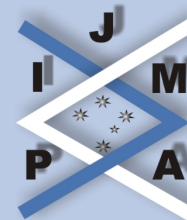
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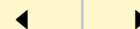
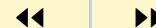
Nisar A. Rather

vol. 9, iss. 4, art. 103, 2008

---

Title Page

Contents



Page 20 of 21

Go Back

Full Screen

Close

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Title Page

Contents



Page 21 of 21

Go Back

Full Screen

Close