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INEQUALITIES INVOLVING MULTIPLIERS FOR MULTIVALENT HARMONIC FUNCTIONS

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ABSTRACT. We introduce inequalities involving multipliers for complex-valued multivalent harmonic functions, using two sequences of positive real numbers. By specializing those sequences, we determine representation theorems, distortion bounds, integral convolutions, convex combinations and neighborhoods for such functions. The theorems presented, in many cases, confirm or generalize various well-known results for corresponding classes of multivalent or univalent harmonic functions.

Key words and phrases: Multivalent harmonic, Multivalent harmonic starlike, Multivalent harmonic convex, Multiplier, Integral convolution, Neighborhood.

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1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. Such functions admit the representation $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. In [5], it was observed that $f = h + \bar{g}$ is locally univalent and sense preserving if and only if $|g'(z)| < |h'(z)|, \ \ z \in \mathbb{D}$.

The study of harmonic functions which are multivalent in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ was initiated by Duren, Hengartner and Laugesen [6]. However, passing from harmonic univalent functions to the harmonic multivalent functions turns out to be quite non-trivial. In view of the argument principle for harmonic functions obtained in [6], the second author and

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Jahangiri [1, 2] introduced and studied certain subclasses of the family $H(m)$, $m \geq 1$, of all m-valent harmonic and orientation preserving functions in $\mathbb{U}$. A function f in $H(m)$ can be expressed as $f = h + \overline{g}$, where h and g are analytic functions of the form

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \qquad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \qquad |b_m| < 1. \tag{1.1}$$

The class $H(1)$ of harmonic univalent functions was studied by Clunie and Sheil-Small [5].

Let $SH(m,\alpha)$, $m \geq 1$ and $0 \leq \alpha < 1$ denote the class of functions $f = h + \overline{g} \in H(m)$ which satisfy the condition

$$\frac{\partial}{\partial \theta}(\arg(f(re^{i\theta}))) \geq m\alpha, \tag{1.2}$$

for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$. A function f in $SH(m,\alpha)$ is called an m-valent harmonic starlike function of order α. Also, let $TH(m,\alpha)$, $m \geq 1$, denote the class of functions $f = h + \overline{g} \in SH(m,\alpha)$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}, \qquad g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \quad |b_m| < 1. \tag{1.3}$$

The class $TH(m,\alpha)$ was studied by second author and Jahangiri in [1, 2]. In particular, they stated the following:

Theorem A. *Let $f = h + \bar{g}$ be given by (1.3). Then f is in $TH(m,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left[\frac{n-1+m(1-\alpha)}{m(1-\alpha)} |a_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)} |b_{n+m-1}| \right] \leq 2, \tag{1.4}$$

where $a_m = 1$ and $m \geq 1$.

Analogous to $TH(m,\alpha)$ is the class $KH(m,\alpha)$ of m-valent harmonic convex functions of order α, $0 \leq \alpha < 1$. More precisely, a function $f = h + \overline{g}$, where h and g are of the form (1.3), is in $KH(m,\alpha)$ if and only if it satisfies the condition

$$\frac{\partial}{\partial \theta}\left(\arg\left(\frac{\partial}{\partial \theta} f(re^{i\theta})\right)\right) \geq m\alpha,$$

for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$.

Theorem B ([4]). *Let $f = h + \bar{g}$ be given by (1.3). Then f is in $KH(m,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{n+m-1}{m^2(1-\alpha)} \left[(n-1+m(1-\alpha))|a_{n+m-1}| + (n-1+m(1+\alpha))|b_{n+m-1}| \right] \leq 2, \tag{1.5}$$

where $a_m = 1$ and $m \geq 1$.

Inequalities (1.4) and (1.5) as well as several such known inequalities in the literature are the motivating forces for introducing a multiplier family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ for $m \geq 1$. A function $f = h + \bar{g}$, where h and g are given by (1.3), is said to be in the multiplier family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ if there exist sequences $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} \left[\frac{c_{n+m-1}}{m} |a_{n+m-1}| + \frac{d_{n+m-1}}{m} |b_{n+m-1}| \right] \leq 2, \qquad c_m = m, d_m |b_m| < m. \tag{1.6}$$

The multipliers $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ provide a transition from multivalent harmonic starlike functions to multivalent harmonic convex functions, including many more subclasses of $H(m)$ and $H(1)$. For example,

$$F_m\left(\left\{\frac{n-1+m(1-\alpha)}{1-\alpha}\right\},\left\{\frac{n-1+m(1+\alpha)}{1-\alpha}\right\}\right)\equiv TH(m,\alpha), \tag{1.7}$$

$$F_m\left(\left\{\frac{(n+m-1)(n-1+m(1-\alpha))}{m(1-\alpha)}\right\},\right. \left.\left\{\frac{(n+m-1)(n-1+m(1+\alpha))}{m(1-\alpha)}\right\}\right)\equiv KH(m,\alpha), \tag{1.8}$$

$$F_m(\{n+m-1\},\{n+m-1\})\equiv TH(m,0):=TH(m), \tag{1.9}$$

$$F_m\left(\left\{\frac{(n+m-1)^2}{m}\right\},\left\{\frac{(n+m-1)^2}{m}\right\}\right)\equiv KH(m,0):=KH(m), \tag{1.10}$$

$$F_1(\{n\},\{n\})\equiv TH(1,0)=TH, \tag{1.11}$$

$$F_1(\{n^2\},\{n^2\})\equiv KH(1,0):=KH, \tag{1.12}$$

$$F_1(\{n^p\},\{n^p\}):=F(\{n^p\},\{n^p\}),\quad p>0, \tag{1.13}$$

$$F_1(\{c_n\},\{d_n\}):=F(\{c_n\},\{d_n\}). \tag{1.14}$$

While (1.7), (1.9) and (1.11) follow immediately from Theorem A, (1.8), (1.10) and (1.12) are consequences of Theorem B. Note that TH and KH in (1.11) and (1.12) were studied in [9] as well as [10]. Also, by letting $m=1,\alpha=0,c_n=d_n=n^p$ for $p>0$ and $b_1=0$, the classes $F_1(\{n^p\},\{n^p\})$ were studied in [8]. Finally, (1.14) follows from (1.6) by setting $m=1$ which was studied in [3].

In this paper, we determine representation theorems, distortion bounds, convolutions, convex combinations and neighborhoods of functions in $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$. As illustrations of our results, the corresponding results for certain families are presented in the corollaries.

2. Main Results

If $(n+m-1)\le c_{n+m-1}$ and $(n+m-1)\le d_{n+m-1}$, then by Theorem A we have

$$F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})\subset TH(m).$$

Consequently, the functions $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ are sense-preserving, harmonic and multivalent in $\mathbb{U}$. We first observe that if

$$f_1(z)=z^m-\sum_{n=2}^{\infty}|a_{1_{(n+m-1)}}|z^{n+m-1}+\sum_{n=1}^{\infty}|b_{1_{(n+m-1)}}|\bar{z}^{n+m-1}$$

and

$$f_2(z)=z^m-\sum_{n=2}^{\infty}|a_{2_{(n+m-1)}}|z^{n+m-1}+\sum_{n=1}^{\infty}|b_{2_{(n+m-1)}}|\bar{z}^{n+m-1}$$

are in $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ and $0 \le \lambda \le 1$, then so is the linear combination $\lambda f_1 + (1-\lambda) f_2$ by (1.6). Therefore, $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ is a convex family.

Next we determine the extreme points of the closed convex hull of the family $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$, denoted by $clcoF_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$.

Theorem 2.1. *A function $f = h + \overline{g}$ is in $clcoF_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ if and only if f has the representation*

$$f(z) = \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z)), \tag{2.1}$$

where

$$(n+m-1) \le c_{n+m-1}, \quad (n+m-1) \le d_{n+m-1}, \quad \lambda_{n+m-1} \ge 0,$$

$$\mu_{n+m-1} \ge 0, \quad \sum_{n=1}^{\infty} (\lambda_{n+m-1} + \mu_{n+m-1}) = 1, \quad h_m(z) = z^m,$$

$$h_{n+m-1}(z) = z^m - \frac{m}{c_{n+m-1}} z^{n+m-1} \quad \text{and} \quad g_{n+m-1}(z) = z^m + \frac{m}{d_{n+m-1}} \overline{z}^{n+m-1}.$$

In particular, the extreme points of $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ are $\{h_{n+m-1}\}$, $\{g_{n+m-1}\}$.

Proof. For functions f of the form (2.1) we have

$$\begin{aligned} f(z) &= \lambda_m h_m(z) + \sum_{n=2}^{\infty} \lambda_{n+m-1} \left(z^m - \frac{m}{c_{n+m-1}} z^{n+m-1} \right) \\ &\quad + \sum_{n=1}^{\infty} \mu_{n+m-1} \left(z^m + \frac{m}{d_{n+m-1}} \overline{z}^{n+m-1} \right) \\ &= z^m - \sum_{n=2}^{\infty} \lambda_{n+m-1} \frac{m}{c_{n+m-1}} z^{n+m-1} + \sum_{n=1}^{\infty} \mu_{n+m-1} \frac{m}{d_{n+m-1}} \overline{z}^{n+m-1}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \lambda_{n+m-1} \frac{m}{c_{n+m-1}} c_{n+m-1} + \sum_{n=1}^{\infty} \mu_{n+m-1} \frac{m}{d_{n+m-1}} d_{n+m-1} \\ &\qquad = \sum_{n=2}^{\infty} m\lambda_{n+m-1} + \sum_{n=1}^{\infty} m\mu_{n+m-1} \le m, \end{aligned}$$

and so $f \in clcoF_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$. Conversely, suppose $f \in clcoF_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$. We set

$$\lambda_{n+m-1} = \frac{c_{n+m-1}}{m} |a_{n+m-1}|, \qquad (n = 2, 3, \dots),$$

$$\mu_{n+m-1} = \frac{d_{n+m-1}}{m} |b_{n+m-1}|, \qquad (n = 1, 2, 3, \dots),$$

and

$$\lambda_m = 1 - \sum_{n=2}^{\infty} \lambda_{n+m-1} - \sum_{n=1}^{\infty} \mu_{n+m-1}.$$

Therefore, by using routine computations, f can be written as

$$f(z) = \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z)).$$

$\square$

In view of (1.7), Theorem 2.1 yields:

Corollary 2.2 ([2]). *A function $f = h + \overline{g}$ is in $clcoTH(m, \alpha)$ if and only if f can be expressed in the form (2.1), where*

$$h_m(z) = z^m, \qquad h_{n+m-1}(z) = z^m - \frac{m(1-\alpha)}{n-1+m(1-\alpha)} z^{n+m-1}, \quad (n = 2, 3, \dots),$$

$$g_{n+m-1}(z) = z^m + \frac{m(1-\alpha)}{n-1+m(1+\alpha)} \overline{z}^{n+m-1}, \quad (n = 1, 2, 3, \dots)$$

and

$$\sum_{n=1}^{\infty} (\lambda_{n+m-1} + \mu_{n+m-1}) = 1, \;\; \lambda_{n+m-1} \geq 0, \quad \mu_{n+m-1} \geq 0.$$

Our next result provides distortion bounds for the functions in $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.3. *Let $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ be increasing sequences of positive real numbers so that*

$$c_{m+1} \leq d_{m+1}, \quad (n+m-1) \leq c_{n+m-1} \quad \textit{and} \quad (n+m-1) \leq d_{n+m-1}$$

for all $n \geq 2$. If $f \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, then

$$(1 - |b_m|)r^m - \left(\frac{m - d_m|b_m|}{c_{m+1}}\right) r^{m+1} \leq |f(z)| \leq (1 + |b_m|)r^m + \left(\frac{m - d_m|b_m|}{c_{m+1}}\right) r^{m+1}.$$

The bounds given above are sharp for the functions

$$f(z) = z^m \pm |b_m|\overline{z}^m + \left(\frac{m - d_m|b_m|}{c_{m+1}}\right) \overline{z}^{m+1}, \quad d_m|b_m| < 1.$$

Proof. Let $f \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$. Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &= \left| z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| \overline{z}^{n+m-1} \right| \\
&\leq r^m + \sum_{n=2}^{\infty} |a_{n+m-1}| r^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| r^{n+m-1} \\
&= (1 + |b_m|)r^m + \sum_{n=2}^{\infty} (|a_{n+m-1}| + |b_{n+m-1}|) r^{n+m-1} \\
&\leq (1 + |b_m|)r^m + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} c_{m+1}(|a_{n+m-1}| + |b_{n+m-1}|) r^{m+1} \\
&\leq (1 + |b_m|)r^m + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} (c_{m+1}|a_{n+m-1}| + d_{m+1}|b_{n+m-1}|) r^{m+1} \\
&\leq (1 + |b_m|)r^m + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} (c_{n+m-1}|a_{n+m-1}| + d_{n+m-1}|b_{n+m-1}|) r^{m+1} \\
&\leq (1 + |b_m|)r^m + \frac{1}{c_{m+1}} (m - d_m|b_m|) r^{m+1}.
\end{aligned}$$

We omit the proof of the left side of the inequality as it is similar to that of the right side. □

Corollary 2.4. *If $f \in TH(m,\alpha)$, then*

$$|f(z)| \leq (1+|b_m|)r^m + \frac{m(1-\alpha-(1+\alpha)|b_m|)}{1+m(1-\alpha)}r^{m+1},$$

and

$$|f(z)| \geq (1-|b_m|)r^m - \frac{m(1-\alpha-(1+\alpha)|b_m|)}{1+m(1-\alpha)}r^{m+1},$$

where $|z| = r < 1$.

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.5. *Let f be as in Theorem 2.3. Then*

$$\left\{w : |w| < \frac{1}{c_{m+1}}(c_{m+1} - m - (c_{m+1} - d_m)|b_m|)\right\} \subset f(\mathbb{U}).$$

Corollary 2.6. *If $f \in TH(m,\alpha)$, then*

$$\left\{w : |w| < \frac{1+(2m\alpha-1)|b_m|}{1+m(1-\alpha)}\right\} \subset f(\mathbb{U}).$$

Remark 2.7. For $\alpha = 0$, the corresponding results in Corollary 2.4 and Corollary 2.6 were also found in [1].

In the next result, we find the convex combinations of the members of the family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.8. *If $(n+m-1) \leq c_{n+m-1}$ and $(n+m-1) \leq d_{n+m-1}$ for all $n+m-1 \geq 2$, then $F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ is closed under convex combinations.*

Proof. Consider

$$f_i(z) = z^m - \sum_{n=2}^{\infty} |a_{i_{n+m-1}}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{i_{n+m-1}}| \bar{z}^{n+m-1}$$

for $i = 1, 2, \ldots$. If $f_i \in F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$ then

$$\sum_{n=2}^{\infty} c_{n+m-1}|a_{i_{n+m-1}}| + \sum_{n=1}^{\infty} d_{n+m-1}|b_{i_{n+m-1}}| \leq m, \qquad i = 1, 2, \ldots. \tag{2.2}$$

For $\sum_{i=1}^{\infty} t_i = 1, \quad 0 \leq t_i \leq 1$, we have

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i |a_{i_{n+m-1}}|\right) z^{n+m-1} + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i |b_{i_{n+m-1}}|\right) \bar{z}^{n+m-1}.$$

In view of the above equality and (2.2), we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} c_{n+m-1}\left|\sum_{i=1}^{\infty} t_i\right| |a_{i_{n+m-1}}| + \sum_{n=1}^{\infty} d_{n+m-1}\left|\sum_{i=1}^{\infty} t_i\right| |b_{i_{n+m-1}}| \\
&= \sum_{i=1}^{\infty} t_i \left\{\sum_{n=2}^{\infty} c_{n+m-1}|a_{i_{n+m-1}}| + \sum_{n=1}^{\infty} d_{n+m-1}|b_{i_{n+m-1}}|\right\} \\
&\leq \sum_{i=1}^{\infty} t_i m = m.
\end{aligned}$$

Hence $\sum_{i=1}^{\infty} t_i f_i \in F_m(\{c_{n+m-1}\},\{d_{n+m-1}\})$, by an application of (1.6). □

In view of relations (1.7) to (1.10), we have the following results:

Corollary 2.9. *The family* $TH(m,\alpha)$, $KH(m,\alpha)$, $TH(m)$ *and* $KH(m)$ *are closed under convex combinations.*

For harmonic functions

$$f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| \bar{z}^{n+m-1} \tag{2.3}$$

and

$$F(z) = z^m - \sum_{n=2}^{\infty} |A_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |B_{n+m-1}| \bar{z}^{n+m-1} \tag{2.4}$$

define the integral convolution of f and F as

$$(f \diamond F)(z) = z^m - \sum_{n=2}^{\infty} \frac{|a_{n+m-1}A_{n+m-1}|}{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} \frac{|b_{n+m-1}B_{n+m-1}|}{n+m-1} \bar{z}^{n+m-1}. \tag{2.5}$$

In the following result, we show the integral convolution property of the class $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.10. *Let* $(n+m-1) \leq c_{n+m-1}$ *and* $(n+m-1) \leq d_{n+m-1}$ *for all* $n+m-1 \geq 2$. *If* f *and* F *are in* $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, *then so is* $f \diamond F$.

Proof. Since $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\}) \subset TH(m)$ and $F \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, it follows that $|A_{n+m-1}| \leq 1$ and $|B_{n+m-1}| \leq 1$. Then $f \diamond F \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ because

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{c_{n+m-1}}{m(n+m-1)} |a_{n+m-1}A_{n+m-1}| + \sum_{n=1}^{\infty} \frac{d_{n+m-1}}{m(n+m-1)} |b_{n+m-1}B_{n+m-1}| \\
&\leq \sum_{n=2}^{\infty} \frac{c_{n+m-1}}{m(n+m-1)} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{d_{n+m-1}}{m(n+m-1)} |b_{n+m-1}| \\
&\leq \sum_{n=2}^{\infty} \frac{c_{n+m-1}}{m} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{d_{n+m-1}}{m} |b_{n+m-1}| \\
&\leq 2.
\end{aligned}$$

□

Corollary 2.11. *If* f *and* F *are in* $TH(m,\alpha)$, $KH(m,\alpha)$, $TH(m)$ *and* $KH(m)$, *then so is* $f \diamond F$.

The $\delta-$*neighborhood* of the functions $f = h + \bar{g}$ in $F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\})$ is defined as the set $N_\delta(f)$ consisting of functions

$$F(z) = z^m + B_m \bar{z}^m + \sum_{n=2}^{\infty} (A_{n+m-1} z^{n+m-1} + B_{n+m-1} \bar{z}^{n+m-1})$$

such that

$$\sum_{n=2}^{\infty} [(n+m-1)(|a_{n+m-1} - A_{n+m-1}| + |b_{n+m-1} - B_{n+m-1}|)] + m|b_m - B_m| \leq \delta, \quad \delta > 0.$$

Our next result guarantees that the functions in a neighborhood of

$$F_m(\{(n+m-1)c_{n+m-1}\},\ \{(n+m-1)d_{n+m-1}\})$$

are multivalent harmonic starlike functions.

Theorem 2.12. *Let* $\{c_{n+m-1}\}$ *and* $\{d_{n+m-1}\}$ *be increasing sequences of real numbers so that* $c_{m+1} \leq d_{m+1},\ (n+m-1) \leq c_{n+m-1}$ *and* $(n+m-1) \leq d_{n+m-1}$ *for all* $n \geq 2$*. If*

$$\delta = \frac{m}{c_{m+1}}(c_{m+1} - 1 - (c_{m+1} - d_m)|b_m|),$$

then

$$N_\delta(F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\})) \subset TH(m).$$

Proof. Suppose

$$f = h + \bar{g} \in F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\}).$$

Let $F = H + \bar{G} \in N_\delta(f)$ where

$$H(z) = z^m + \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1} \qquad \text{and} \qquad G(z) = \sum_{n=1}^{\infty} B_{n+m-1} z^{n+m-1}.$$

We need to show that $F \in TH(m)$. It suffices to show that F satisfies the condition

$$M(F) := \sum_{n=2}^{\infty}(n+m-1)(|A_{n+m-1}| + |B_{n+m-1}|) + m|B_m| \leq m$$

Note that

$$\begin{aligned} M(F) &\leq \sum_{n=2}^{\infty}(n+m-1)[|A_{n+m-1} - a_{n+m-1}| + |B_{n+m-1} - b_{n+m-1}|] + m|B_m - b_m| \\ &\qquad + \sum_{n=2}^{\infty}(n+m-1)(|a_{n+m-1}| + |b_{n+m-1}|) + m|b_m| \\ &\leq \delta + m|b_m| + \sum_{n=2}^{\infty}(n+m-1)(|a_{n+m-1}| + |b_{n+m-1}|) \\ &= \delta + m|b_m| + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} \big(c_{m+1}(n+m-1)|a_{n+m-1}| \\ &\qquad + c_{m+1}(n+m-1)|b_{n+m-1}|\big) \\ &\leq \delta + m|b_m| + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} \big(\,(n+m-1)c_{n+m-1}|a_{n+m-1}| \\ &\qquad + (n+m-1)d_{n+m-1}|b_{n+m-1}|\big) \\ &\leq \delta + m|b_m| + \frac{1}{c_{m+1}}(m(1 - d_m|b_m|)). \end{aligned}$$

But, the last expression is never greater than m provided that

$$\delta \leq m - m|b_m| - \frac{1}{c_{m+1}}(m(1 - d_m|b_m|)) = \frac{m}{c_{m+1}}(c_{m+1} - 1 - (c_{m+1} - d_m)|b_m|).$$

□

Corollary 2.13. *If*

$$\delta = \frac{m - (m - 2m^2\alpha)|b_m|}{1 + m(1-\alpha)},$$

then $N_\delta(KH(m,\alpha)) \subset TH(m)$.

Letting $\alpha = 0$ and $m = 1$, Corollary 2.13 yields the following interesting result.

Corollary 2.14. $N_{\frac{1}{2}(1-|b_1|)}(KH) \subset TH.$

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