journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 9 (2008), Issue 4, Article 110, 10 pp.

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TWO-DIMENSIONAL SUNOUCHI OPERATOR WITH RESPECT TO VILENKIN-LIKE SYSTEMS

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Received 05 May, 2007; accepted 19 October, 2008
Communicated by S.S. Dragomir

ABSTRACT. In this paper two-dimensional Vilenkin-like systems will be investigated. We prove the Sunouchi operator is bounded from H^q to L^q for $(2/3 < q \leq 1)$. As a consequence, we prove the Sunouchi operator is L^s bounded for $1 < s < \infty$ and of weak type $(H^\natural, L^1)$.

Key words and phrases: Sunouchi operator, Vilenkin-like systems.

2000 *Mathematics Subject Classification.* 42C10.

1. INTRODUCTION

The operator U (called the Sunouchi operator) was first introduced and investigated by Sunouchi [1], [2] in Walsh-Fourier analysis.He showed a characterization for the L^p spaces for $p > 1$ by means of U, since this characterization fails to hold for $p = 1$. It was of interest to investigate the boundedness of U on a Hardy space. In [3] Simon showed that U is a sublinear bounded map from the dyadic Hardy space H^1 into L^1.

The Vilenkin analogue of the Sunouchi operator was given by Gát [4], [5]. He investigated the boundedness of U from (Vilenkin) H^1 into L^1 and proved that if a Vilenkin group has an unbounded structure and H^1 is defined by means of the usual maximal function, then U is not bounded. Furthermore, when they considered a modified H^1 space (introduced by Simon [6]), then a necessary and sufficient condition could be given for a Vilenkin group that $U : H^1 \to L^1$ be bounded. All Vilenkin groups with bounded structure and certain groups without this boundedness property satisfy the condition given by Gát. Thus, in the so-called bounded case, the (H^1, L^1) -boundedness of U remains true also for Vilenkin system. In [7] Simon extended

Supported by Foundation of Hubei Scientific Committee under grant No.B20081102.

The author thanks the referees for their helpful advice.

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this result, by showing the (H^q, L^q)-boundedness of U for all $0 < q \leq 1$. Moreover, the equivalence

$$\|f\|_{H^q} \sim \|Uf\|_q \quad \left(\frac{1}{2} < q \leq 1\right)$$

was also obtained for f with mean value zero.

In this paper we consider a two-dimensional case with respect to generalized Vilenkin-like systems.

2. PRELIMINARIES AND NOTATIONS

In this section, we introduce important definitions and notations. Furthermore, we formulate some known results with respect to Vilenkin-like systems, which play a basic role in further investigations. For details, see [8] by Vilenkin and [9] by Schipp, Wade, Simon and Pál.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \dots, \})$ be a sequence of integers, each of them not less than 2. Denote by Z_{m_k} the m_k -th cyclic group $(k \in \mathbb{N})$. That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, where the group operator is the mod m_k addition and every subset is open. The Harr measure on Z_{m_k} is given such that $\mu(\{j\}) = \frac{1}{m_k}$ $(j \in Z_{m_k}, k \in \mathbb{N})$.

Let G_m denote the complete direct product of Z_{m_k}'s equipped with product topology and product measure μ, then G_m forms a compact Abelian group with Haar measure 1. The elements of G_m are sequences of the form $(x_0, x_1, \dots, x_k, \dots)$, where $x_k \in Z_{m_k}$ for every $k \in \mathbb{N}$ and the topology of the group G_m is completely determined by the sets

$$I_n(0) := \{(x_0, x_1, \dots, x_k, \dots) \in G_m : x_k = 0 \ (k = 0, \dots, n-1)\}$$

$(I_0(0) := G_m)$. Let $I_n(x) := I_n(0) + x$ $(n \in \mathbb{N})$; $M_0 := 1$ and $M_{k+1} := m_k M_k$ for $k \in \mathbb{N}$, the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. The sequence $(n_0, n_1, \dots)$ is called the expansion of n with respect to m. We often use the following notations: $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$.

Let $\hat{G}_m := \{\psi_n : n \in \mathbb{N}\}$ denote the character group of G_m. We enumerate the elements of $\hat{G}_m$ as follows. For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp\left(2\psi\imath \frac{x_k}{m_k}\right) \quad (x \in G_m, \imath : \sqrt{-1}, k \in \mathbb{N}).$$

It is known for $x \in G_m, n \in \mathbb{N}$ that

$$\sum_{i=0}^{m_n - 1} r_n^i(x) = \begin{cases} 0, & \text{if } x_n \neq 0; \\ m_n, & \text{if } x_n = 0. \end{cases} \tag{2.1}$$

Now we define the ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k} \quad (n \in \mathbb{N}).$$

$\hat{G}_m$ is a complete orthonormal system with respect to μ.

G. Gát introduced the so-called Vilenkin-like (or $\psi\alpha$) system. Let functions $\alpha_n, \alpha_j^k : G_m \to \mathcal{C}$ $(n, j, k \in \mathbb{N})$ satisfy:

i) α_j^k is measurable with respect to Σ_j (i.e. α_j^k depends only on $x_0, x_1, \dots, x_{j-1}$, $j, k \in \mathbb{N}$);

ii) $|\alpha_j^k| = \alpha_j^k(0) = \alpha_0^k = \alpha_j^0 = 1$ $(j, k \in \mathbb{N})$;

iii) $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{n^{(j)}}$ $(n \in \mathbb{N})$.

Let $\chi_n := \psi_n \alpha_n$ $(n \in \mathbb{N})$. The system $\chi := \{\chi_n : n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system (see [10] and [13] for examples).

(1) If $\alpha_j^k = 1$ for each $k, j \in \mathbb{N}$, then we have the "ordinary" Vilenkin systems.
(2) If $m_j = 2$ for all $j \in \mathbb{N}$ and $\alpha_j^{n^{(j)}} = (\beta_j)^{n_j}$, where

$$\beta_j(x) = \exp\left(2\pi\iota\left(\frac{x_{j-1}}{2^2} + \cdots + \frac{x_0}{2^{j+1}}\right)\right) \quad (n, j \in \mathbb{N}, x \in G_m),$$

then we have the character system of the group of 2-adic integers.
(3) If

$$\chi_n(x) := \exp\left(2\pi\iota\left(\sum_{j=0}^{\infty} \frac{n_j}{M_{j+1}} \sum_{j=0}^{\infty} x_j M_j\right)\right) \quad (x \in G_m, n \in \mathbb{N}),$$

then we have a Vilenklin-like system which is useful in the approximation of limit periodic almost even arithmetical functions.

In [10] Gát proved that a Vilenkin-like system is orthonormal and complete in $L^1(G_m)$. Define the Fourier coefficients, the Dirichlet kernels, and Fejér kernels with respect to the Vilenkin-like system χ as follows:

$$\hat{f}^\chi(n) = \hat{f}(n) := \int_{G_m} f\bar{\chi}_n, \quad \hat{f}^\chi(0) := \int_{G_m} f \qquad (f \in L^1(G_m));$$

$$D_n^\chi(y,x) = D_n(y,x) := \sum_{k=0}^{n-1} \chi_n(y)\bar{\chi}_n(x);$$

$$K_n^\chi(y,x) = K_n(y,x) := \frac{1}{n}\sum_{k=0}^{n-1} D_n^\chi(y,x);$$

$$K_{h,H}^\chi(y,x) = K_{h,H}(y,x) := \sum_{j=h}^{h+H-1} D_j^\chi(y,x),$$

where the bar means complex conjugation.

In [10] Gát also proved the following expression of the Dirichlet kernel functions.

$$D_{M_n}^\chi(y,x) = D_{M_n}^\psi(y-x) = \begin{cases} M_n, & \text{if } y-x \in I_n \\ 0, & \text{if } y-x \in G_m \backslash I_n. \end{cases} \tag{2.2}$$

Moreover,

$$\begin{aligned} D_n^\chi(y,x) &= \alpha_n(y)\bar{\alpha}_n(x)D_n^\psi(y-x) \\ &= \chi_n(y)\bar{\chi}_n(x)\left(\sum_{j=0}^{\infty} D_{M_j}(y-x) \sum_{k=m_j-n_j}^{m_j-1} r_j^k(y-x)\right) \\ &\quad (n \in \mathbb{P} := \mathbb{N}\backslash\{0\},\ y, x \in G_m), \end{aligned}$$

where the system ψ is the "ordinary" Vilenkin system.

If $\tilde{m} = (\tilde{m}_n, n \in \mathbb{N})$ is also a generating sequence then we consider the Vilenkin group $G_{\tilde{m}}$ as well. We write $\tilde{M}_n$ instead of M_n. Let $G := G_m \times G_{\tilde{m}}$ and

$$\chi_{k,\,l}(x,y) = \chi_k(x)\chi_l(y) \qquad (k, l \in \mathbb{N},\ x \in G_m,\ y \in G_{\tilde{m}})$$

be the two-parameter Vilenkin groups and Vilenkin systems, respectively.

The symbol L^p $(0 < p \leq \infty)$ will denote the usual Lebesgue space of complex-valued functions f defined on G with the norm (or quasinorm)

$$\|f\|_p := \left(\int_G |f|^p\right)^{\frac{1}{p}} \quad (0 < p < \infty), \quad \|f\|_\infty := \operatorname{ess\,sup}|f|.$$

If $f \in L^1$, then $\hat{f}(k,l) := \int_G f\overline{\chi_{k,l}}$ $(k, L \in \mathbb{N})$ is the usual Fourier coefficient of f. Let $S_{n,l}f$ $(n, l \in \mathbb{N})$ be the (n,l)-th rectangular partial sum of f:

$$S_{n,l}f := \sum_{k=0}^{n-1}\sum_{j=0}^{l-1} \hat{f}(k,j)\chi_{k,j}.$$

The so-called (martingale) maximal function of f is given by

$$f^*(x,y) = \sup_{n,\ l} M_n\tilde{M}_l \left|\int_{I_n(x)}\int_{I_l(y)} f\right| \qquad (x \in G_m,\ y \in G_{\tilde{m}}).$$

Furthermore, let $f^\natural$ be the hybrid maximal function of f defined by

$$f^\natural(x,y) := \sup_n M_n \left|\int_{I_n(x)} f(t,y)dt\right| \qquad (x \in G_m,\ y \in G_{\tilde{m}}).$$

Define the Hardy space $H^p(G_m \times G_{\tilde{m}})$ for $0 < p < \infty$ as the space of functions f for which

$$\|f\|_{H^p} := \|f^*\|_p < \infty.$$

Then $\|f\|_{H^p}$ is equivalent to $\|Qf\|_p$, where Qf is the quadratic variation of f:

$$Qf := \left(\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} |\Delta_{n,l}f|^2\right)^{\frac{1}{2}}$$
$$:= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \left|S_{M_n,\tilde{M}_l}f - S_{M_n,\tilde{M}_{l-1}}f - S_{M_{n-1},\tilde{M}_l}f + S_{M_{n-1},\tilde{M}_{l-1}}f\right|^2\right)^{\frac{1}{2}}$$
$$S_{M_n,\tilde{M}_{-1}}f := S_{M_{-1},\tilde{M}_l}f := S_{M_{-1},\tilde{M}_{-1}}f := 0 \qquad (n, l \in \mathbb{N}).$$

Let $H^\natural$ be the set of functions f such that

$$\|f\|_{H^\natural} := \|f^\natural\|_1 < \infty.$$

In [11] Weisz defined the two-dimensional Sunouchi operator as follows:

$$Uf := \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |S_{2^n,2^m}f - S_{2^n}\sigma_{2^m}f - \sigma_{2^n}S_{2^m}f + \sigma_{2^n}\sigma_{2^m}f|^2\right)^{\frac{1}{2}}$$

where σf is the Cesàro means of the Walsh Fourier series of $f \in L^1$. Now we extend the definition to the two-dimensional Vilenkin-like systems as follows:

$$Uf := \left(\sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \left|\sum_{j=1}^{M_{n+1}-1}\sum_{k=1}^{\tilde{M}_{s+1}-1} \frac{jk}{M_{n+1}\tilde{M}_{s+1}}\hat{f}(j,k)\chi_{j,k}\right|^2\right)^{\frac{1}{2}} \qquad (f \in L^1).$$

If $\alpha = (\alpha_n,\ n \in \mathbb{N}),\ \beta = (\beta_n,\ n \in \mathbb{N})$ are bounded sequences of complex numbers, then let

$$T_{\alpha,\beta}f := \sup_{n,l} \sum_{i=0}^{M_n-1}\sum_{j=0}^{\tilde{M}_l-1} \alpha_n\beta_k\hat{f}(n,k)\chi_{n,k}$$

be defined at least on L^2.

Moreover, let $\alpha_j := jM_l^{-1}$ $(l \in \mathbb{N}, j = M_l, \ldots, M_{l+1}-1)$ and $\beta_k := k\tilde{M}_t^{-1}$ $(t \in \mathbb{N}, k = \tilde{M}_t, \ldots, \tilde{M}_{t+1}-1)$ then

$$Uf = \left(\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left| \sum_{l=0}^{n} \sum_{t=0}^{s} M_l \tilde{M}_t \Delta_{l+1,t+1}(T_{\alpha,\beta} f) \right|^2 \right)^{\frac{1}{2}}.$$

In this paper we assume the sequences $m,\ \tilde{m}$ are bounded. In the investigations of some operators defined on Hardy spaces, the concept of a q-atom is very useful. The function a is called a q-atom if either a is identically equal to 1 or there exist intervals $I_n(\tau) \subset G_m,\ I_L(\gamma) \subset G_{\tilde{m}}$ $(N,\ L \in \mathbb{N},\ \tau \in G_m,\ \gamma \in G_{\tilde{m}})$ such that

$$\begin{aligned} &i) \quad a(x,y) = 0 \text{ if } (x,y) \in G \backslash (I_N(\tau) \times I_L(\gamma)), \\ &ii) \quad \|a\|_2 \le \mu(I_N(\tau) \times I_L(\gamma))^{\frac{1}{2}-\frac{1}{q}}, \\ &iii) \quad \int_{G_m} a(t,y)dt = \int_{G_{\tilde{m}}} a(x,u)du = 0 \text{ if } x \in G_m,\ y \in G_{\tilde{m}}. \end{aligned}$$

Lemma 2.1 ([1]). *Let T be an operator defined at least on L_2 and assume that T is L_2 bounded. If there exists $\delta > 0$ such that for all q-atoms a with support $I_N(\tau) \times I_L(\gamma)$ and for all $r \in \mathbb{N}$, we have*

$$\int_{G \backslash I_{N-r}(\tau) \times I_{L-r}(\gamma)} |Ta|^q \le C_q 2^{-\delta r},$$

then T is bounded from H_q to L_q for all $0 < q \le 1$.

Lemma 2.2. *Let $\frac{2}{3} < q \le 1$. Then there exist $\delta > 0$ and a constant C_q depending only on q such that for $N, L, r \in \mathbb{N}$*

$$M_N^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \backslash I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\overline{\chi}_k(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \le C_q 2^{-\delta r}.$$

Proof. For $n \in \mathbb{N}, n \ge N$, we have

$$\begin{aligned} M_n K_{M_n}(x,t) &= \sum_{i=0}^{M_n-2} \chi_i(x)\bar{\chi}_i(t) \sum_{k=i+1}^{M_n-1} 1 \\ &= \sum_{i=0}^{M_n-2} (M_n - i - 1)\chi_i(x)\bar{\chi}_i(t) \\ &= (M_n - 1)D_{M_n-1}(x,t) - \sum_{i=0}^{M_n-1} i\chi_i(x)\bar{\chi}_i(t). \end{aligned}$$

This follows

$$\begin{aligned} \sum_{i=M_n}^{M_{n+1}-1} \frac{i\chi_i(x)\bar{\chi}_i(t)}{M_n} &= m_n(D_{M_{n+1}}(x,t) - K_{M_{n+1}}(x,t)) \\ &\quad - (D_{M_n}(x,t) - K_{M_n}(x,t)) - \frac{D_{M_{n+1}}(x,t) - D_{M_n}(x,t)}{M_n}. \end{aligned}$$

If $x \in G_m \backslash I_{N-r},\ t \in I_N$, then there exists u $(0 \le u \le N-r-1)$ such that $x \in I_u \backslash I_{u+1}$. Since $x - t \in I_u \backslash I_{u+1}$, we have $D_{M_k}(x,t) = 0$ for all $(k \ge u+1)$. Suppose that $s > u$. From the definitions of the function α_n and the Fejér kernel, we have, if $x \in I_u(t) \backslash I_{u+1}(t)$,

$$\begin{aligned} K_{n^{(s)},\, M_s}(x,t) = & \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} \left(\sum_{j=0}^{u-1} k_j M_j \right) \chi_k(x)\bar{\chi}_k(t) \\ & \qquad + \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} M_u \sum_{p=m_u-k_u}^{m_u-1} r_t^p(x-t)\chi_k(x)\bar{\chi}_k(t) \\ =: & \sum^{1} + \sum^{2}, \end{aligned}$$

where

$$\begin{aligned} \sum^{1} = & \sum_{k_{s-1}=0}^{m_{s-1}-1} \cdots \sum_{k_{u+1}=0}^{m_{u+1}-1} \sum_{k_{u-1}=0}^{m_{u-1}-1} \cdots \sum_{k_0=0}^{m_0-1} \left(\sum_{j=0}^{t-1} k_j M_j \right) \\ & \qquad \cdot \prod_{l=u+1}^{\infty} r_l^{k_l}(x-t)\alpha_l^{k^{(l)}}(x)\bar{\alpha}_l^{k^{(l)}}(t) \sum_{k_u=0}^{m_u-1} r_u^{k_u}(x-t) \\ = & \sum_{k_u=0}^{m_u-1} r_u^{k_u}(x-t)\phi(x,t), \end{aligned}$$

and the function ϕ does not depend on k_t. Consequently, $\sum^1 = 0$ (see [12]).

Since the sequence m is bounded, we have

$$\begin{aligned} \int_{I_N} \left| \sum^{2} \right|^2 dt \le & \ CM_u^2 \sum_{p=0}^{m_u-1} \int_{I_N} \sum_{k,\ l=0; k_u=m_u=p}^{M_s-1} \chi_{n^{(s)}+k}(t)\bar{\chi}_{n^{(s)}+l}(t)\bar{\chi}_{n^{(s)}+k}(x)\chi_{n^{(s)}+l}(x)dt \\ \le & \ CM_u^2 \frac{1}{M_N} M_s M_u. \end{aligned}$$

Recall that $k^{(u+1)} \neq l^{(u+1)}$ implies

$$\int_{I_N} \chi_{n^s+k}(x)\bar{\chi}_{n^{(s)}+l}(x)dx = 0.$$

If $s \le u$, then $|K_{n^{(s)},\, M_s}(x,t)| \le CM_u M_s$. Then

$$\begin{aligned} & M_N^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \backslash I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\bar{\chi}_i(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \\ & \le M_N^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \backslash I_{N-r}} \left(\int_{I_N} C(|D_{M_{n+1}}(x,t) - K_{M_{n+1}}(x,t)|^2 \right. \\ & \qquad \left. + \left[|D_{M_n}(x,t) - K_{M_n}(x,t)| + \left| \frac{D_{M_{n+1}}(x,t) - D_{M_n}(x,t)}{M_n} \right| \right]^2 dt \right)^{\frac{q}{2}} dx \\ & = M_N^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \backslash I_{N-r}} \left(\int_{I_N} C(|K_{M_{n+1}}(x,t)|^2 + |K_{M_n}(x,t)|^2)dt \right)^{\frac{q}{2}} dx \end{aligned}$$

$$\le C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_{n+1}} \sum_{s=0}^{n+1} \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\int_{I_N} |K_{n^{(s+1)}+jM_s,\, M_s}(x,t)|^2 dt \right)^{\frac{q}{2}} dx$$

$$+ C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_n} \sum_{s=0}^{n} \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\int_{I_N} |K_{n^{(s+1)}+jM_s,\, M_s}(x,t)|^2 dt \right)^{\frac{q}{2}} dx$$

$$\le C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_{n+1}} \sum_{s=0}^{n+1} \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\frac{M_u^3 M_s}{M_N} \right)^{\frac{q}{2}} dx$$

$$+ C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_n} \sum_{s=0}^{n} \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\frac{M_u^3 M_s}{M_N} \right)^{\frac{q}{2}} dx$$

$$\le C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} M_u^{3q/2-1} M_n^{-q/2} M_N^{-q/2}$$

$$\le C_q M_N^{1-q/2} M_{N-r-1}^{3q/2-1} M_N^{-q} = C_q (m_{N-r} \cdots m_{N-1})^{-(3q/2-1)} \le C_q 2^{-\delta r} \quad (\delta = 3q/2 - 1 > 0).$$

□

Theorem 2.3. *Let $\frac{2}{3} < q \le 1$. Then there exists a constant C_q such that*

$$\|Uf\|_q \le C_q \|f\|_{H^q} \quad (\forall f \in H^q(G_m \times G_{\tilde{m}})).$$

Proof. Let a be a q-atom. It can be assumed that the support of a is $I_N \times I_L$ for some $N, L \in \mathbb{N}$, that is

$$\|a\|_2 \le (M_N P_L')^{\frac{1}{q}-\frac{1}{2}} \text{ and } \int_{I_L} a(x,t)dt = \int_{I_N} a(u,y)du = 0 \quad \text{for all } x \in G_m, y \in G_{\tilde{m}}.$$

This last property implies that

$$\hat{a}(i,j) = 0 \text{ if } i = 0, \ldots, M_N - 1 \text{ or } j = 0, \ldots, \tilde{M}_L - 1.$$

Let α and β as above. Then from the Cauchy inequality we have

$$T_{\alpha,\beta} a(x,y)$$

$$\le \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \int_{I_N} \int_{J_L} |a(t,u)| \sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n} \chi_k(x) \bar{\chi}_k(t) \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) | dt du$$

$$\le \|a\|_2 \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \left(\int_{I_N} \int_{J_L} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n} \chi_k(x) \bar{\chi}_i(t) \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) \right|^2 dt du \right)^{\frac{1}{2}}. \tag{2.3}$$

First we will show $T_{\alpha,\beta}$ is q-quasi local. Let $r \in \mathbb{N}$ and define the sets X_i $(i = 1, 2, 3, 4)$ as follows:

$$X_1 := (G_m \setminus I_{N-r}) \times I_L, \quad X_2 := (G_m \setminus I_{N-r}) \times (G_{\tilde{m}} \setminus I_L),$$
$$X_3 := I_N \times (G_{\tilde{m}} \setminus I_{L-r}), \quad X_4 := (G_m \setminus I_N) \times (G_{\tilde{m}} \setminus I_{L-r}).$$

It is clear that

$$\int_{(G \setminus I_{N-r} \times I_{L-r})} (T_{\alpha,\beta} a)^q \le \sum_{i=1}^{4} \int_{X_i} (T_{\alpha,\beta} a)^q.$$

To estimate the integral over X_1, we have

$$\begin{aligned}
&\int_{X_1} (T_{\alpha,\beta}a)^q(x,y)dxdy \\
&\leq |I_L|^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m\setminus I_{N-r}} \left(\int_{I_L} \left(\int_{I_n} \left|\sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n}\chi_k(x)\bar{\chi}_k(t)\right| \right.\right. \\
&\qquad \left.\left. \times \sup \int_{I_L} a(t,u) \sum_{j=L+1}^{l} \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j}\chi_l(y)\bar{\chi}_l(u)|du|dt\right)^2 dy\right)^{\frac{q}{2}} dx \\
&\leq |I_L|^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m\setminus I_{N-r}} \left(\int_{I_N} \left|\sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n}\chi_k(x)\bar{\chi}_k(t)\right|^2 dt\right)^{\frac{q}{2}} dx \\
&\qquad \times \left(\int_{I_N}\int_{J_L} |a(t,y)|^2 dydt\right)^{\frac{q}{2}}.
\end{aligned}$$

From the definition of q-atoms and Lemma 2.2, we have

$$\begin{aligned}
&\int_{X_1} (T_{\alpha,\beta}a)^q(x,y)dxdy \\
&\leq \|a\|_2^q |I_L|^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m\setminus I_{N-r}} \left(\int_{I_N} \left|\sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\bar{\chi}_k(t)}{M_n}\right|^2 dt\right)^{\frac{q}{2}} dx \\
&\leq C_q M_N^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m\setminus I_{N-r}} \left(\int_{I_N} \left|\sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\bar{\chi}_k(t)}{M_n}\right|^2 dt\right)^{\frac{q}{2}} dx \\
&\leq C_q 2^{-\delta r}. \qquad (2.4)
\end{aligned}$$

In a similar way, we have

$$\int_{X_3} (T_{\alpha,\beta}a)^q(x,y)dxdy \leq C_q 2^{-\delta r}. \qquad (2.5)$$

On the set X_2, by inequality (2.3) we have

$$\begin{aligned}
&\int_{X_3} (T_{\alpha,\beta}a)^q(x,y)dxdy \\
&\leq \|a\|_2^q \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \int_{G_m\setminus I_{N-r}} \int_{G_{\tilde{m}}\setminus I_l} \\
&\qquad \left(\int_{I_N}\int_{J_L} \left|\sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\bar{\chi}_k(t)}{M_n} \sum_{l=M_j-1}^{M_j-1} \frac{l}{M_j}\chi_l(y)\bar{\chi}_l(u)\right|^2 dtdu\right)^{\frac{q}{2}} dxdy
\end{aligned}$$

$$\leq (M_N P_L)^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \int_{G_m \setminus I_{N-r}} \int_{G_{\tilde{m}} \setminus I_l}$$

$$\left(\int_{I_N} \int_{J_L} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\bar{\chi}_k(t)}{M_n} \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y)\bar{\chi}_l(u) \right|^2 dtdu \right)^{\frac{q}{2}} dxdy$$

$$\leq M_N^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k\chi_k(x)\bar{\chi}_k(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx$$

$$\leq C_q 2^{-\delta r} (\tilde{M}_L)^{1-\frac{q}{2}} \sum_{j=L+1}^{\infty} \int_{G_{\tilde{m}} \setminus J_L} \left(\int_{I_L} \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y)\bar{\chi}_l(u)|^2 du)^{\frac{q}{2}} dy \right)$$

$$\leq C_q 2^{-\delta r}.$$

An analogous estimate with X_4 instead of X_2 can be obtained using a similar argument and these prove that the operator $T_{\alpha,\beta}$ is q-quasi local. By Parseval's equality, it is clear that the operator $T_{\alpha,\beta}$ is L^2 bounded. Since

$$Uf = \left(\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left| \sum_{j=1}^{M_{n+1}-1} \sum_{k=1}^{\tilde{M}_{s+1}-1} \frac{jk}{M_{n+1}\tilde{M}_{s+1}} \hat{f}(j,k)\chi_{j,k} \right|^2 \right)^{\frac{1}{2}} \leq CQ(T_{\alpha,\beta}f),$$

where the operator Q is a two-dimensional quadratic variation of f. By Lemma 2.1, we have

$$\|Uf\|_q \leq C_q \|Q(T_{\alpha,\beta}f)\|_q \leq C_q \|T_{\alpha,\beta}f\|_{H_q} \leq C_q \|f\|_{H_q}.$$

□

Applying known theorems on the interpolation of operators and a duality argument gives the following:

Theorem 2.4. *The operator U is $L^s \to L^s$ bounded and of weak type $(H^\natural, L^1)$, i.e., there exists a constant C such that for all $\delta > 0$ and $f \in H^\natural$ we have*

$$\mu\{(x,y) \in G : |Uf(x,y)| > \delta\} \leq C \frac{\|f\|_{H^\natural}}{\delta}.$$

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