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A CRITERION FOR p-VALENTLY STARLIKENESS

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Abstract

It is the purpose of the present paper to obtain some sufficient conditions for p- valently starlikeness for a certain class of functions which are analytic in the open unit disk E.

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1. Introduction

Let $A(p)$ be the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in $E = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f(z) \in A(p)$ is said to be p-valently starlike if and only if

$$\operatorname{Re}\left\{ z\frac{f'(z)}{f(z)} \right\} > 0 \quad (z \in E).$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p-valently in E (see, e.g., Goodman [1]).

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f(z)$ of the form (1.1) is said to be $\alpha-$convex in E if it is regular,

$$\frac{f(z)}{z} f'(z) \neq 0,$$

and

$$\operatorname{Re}\left(\alpha \left(1 + z\frac{f''(z)}{f'(z)} \right) + (1-\alpha) z \frac{f'(z)}{f(z)} \right) \geqslant 0 \tag{1.2}$$

for all z in E. The set of all such functions is denoted by $\alpha - CV$, where α is a real number. Of course, if $\alpha = 1$, then an $\alpha-$convex function is convex; and if $\alpha = 0$, an $\alpha-$convex function is starlike. Thus the sets $\alpha - CV$ give a "continuous" passage from convex functions to starlike functions. Sakaguchi considers functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

where p is a positive integer, and he imposes the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right\} \geqslant 0 \tag{1.3}$$

for z in E. He proved that if $k = -1$, there is only one function that satisfies (1.3), namely $f(z) \equiv z^p$. If $-1 < k \leqslant 0$,then $f(z)$ is p-valent convex; and if $0 < k$, then $f(z)$ is p-valent starlike. We can pass from (1.3) back to (1.2) if we divide by $1 + k > 0$ and set $\alpha = \frac{1}{1+k}$ [6]. We denote by $S(p, k)$ the subclass $A(p)$ consisting of functions which satisfy the condition (1.3).

Obradovic and Owa [7] have obtained a sufficient condition for starlikeness of $f(z) \in A(1)$ which satisfies a certain condition for the modulus of

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}},$$

we recall their result as:

Theorem 1.1. *If $f(z) \in A(1)$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < K \left|\frac{zf'(z)}{f(z)}\right| \quad (z \in E),$$

where $K = 1.2849\ldots$, then $f(z) \in S(1)$.

Nunokawa [4] improved Theorem 1.1 by proving

Theorem 1.2. *If $f(z) \in A(p)$, and if*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \left|\frac{zf'(z)}{f(z)}\right| \frac{1}{p} \log(4e^{p-1}) \quad (z \in E),$$

then $f(z) \in S(p)$.

2. Preliminaries

In order to obtain our main result, we need the following lemma attributed to Jack [2] (given also by Miller and Mocanu [3]).

Lemma 2.1. *Let* $w(z)$ *be analytic in* E *with* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value in the circle* $|z| = r < 1$ *at a point* z_0*, then we can write* $z_0 w'(z_0) = kw(z_0)$*, where* k *is a real number and* $k \geq 1$*.*

Making use of Lemma 2.1,we first prove

Lemma 2.2. *Let* $q(z)$ *be analytic in* E *with* $q(0) = p$ *and suppose that*

$$\operatorname{Re}\left\{\frac{zq'(z)}{[q(z)]^2}\right\} < \frac{1}{p(\lambda+1)} \quad (z \in E, 0 \leqslant \lambda \leqslant 1), \tag{2.1}$$

then $\operatorname{Re}\{q(z)\} > 0$ in E.

Proof. Let us put

$$q(z) = p\left\{\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+w(z)}{1-w(z)} + \left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-w(z)}{1+w(z)}\right\},$$

where $0 \leqslant \lambda \leqslant 1$.

Then $w(z)$ is analytic in E with $w(0) = 0$ and by an easy calculation, we have

$$1 + z\frac{q'(z)}{[q(z)]^2} = 1 + \frac{2}{p}\cdot\frac{(\lambda w^2(z) + 2w(z) + \lambda)zw'(z)}{(w^2(z) + 2\lambda w(z) + 1)^2}.$$

If we suppose that there exists a point $z_0 \in E$ such that $\max_{|z|\leqslant|z_0|} |w(z)| = |w(z_0)| = 1$, then, from Lemma 2.1, we have $z_0 w'(z_0) = kw(z_0), \quad (k \geqslant 1)$.

Putting $w(z_0) = e^{i\theta}$, we find that

$$\begin{aligned}
z_0 \frac{q'(z_0)}{[q(z_0)]^2} &= \frac{2}{p} \cdot \frac{\lambda w^2(z_0) w'(z_0) z_0 + 2w(z_0) w'(z_0) z_0 + \lambda w'(z_0) z_0}{[w^2(z_0) + 2\lambda w(z_0) + 1]^2} \\
&= \frac{2k}{p} \cdot \frac{\lambda e^{3i\theta} + 2e^{2i\theta} + \lambda e^{i\theta}}{(e^{2i\theta} + 2\lambda e^{i\theta} + 1)^2} \\
&= \frac{2k}{p} \cdot \frac{\left(\lambda e^{3i\theta} + 2e^{2i\theta} + \lambda e^{i\theta}\right)}{(e^{2i\theta} + 2\lambda e^{i\theta} + 1)^2} \cdot \frac{\left(e^{-2i\theta} + 2\lambda e^{-i\theta} + 1\right)^2}{\left(e^{-2i\theta} + 2\lambda e^{-i\theta} + 1\right)^2} \\
&= \frac{k}{p} \cdot \frac{\lambda \cos 3\theta + (4\lambda^2 + 2)\cos 2\theta + (11\lambda + 4\lambda^3)\cos\theta + (8\lambda^2 + 2)}{4(\lambda + \cos\theta)^4} \\
&= \frac{k}{p} \cdot \frac{(1 + \lambda\cos\theta)(\lambda + \cos\theta)^2}{(\lambda + \cos\theta)^4} \\
&= \frac{k}{p} \cdot \frac{1 + \lambda\cos\theta}{(\lambda + \cos\theta)^2},
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{Re}\left\{ z_0 \frac{q'(z_0)}{[q(z_0)]^2} \right\} &= \frac{k}{p} \cdot \frac{1 + \lambda\cos\theta}{(\lambda + \cos\theta)^2} = \frac{k}{p} \cdot \frac{\lambda^2 + \lambda\cos\theta + 1 - \lambda^2}{(\lambda + \cos\theta)^2} \\
&= \frac{k}{p} \left\{ \frac{\lambda}{(\lambda + \cos\theta)} + \frac{1 - \lambda^2}{(\lambda + \cos\theta)^2} \right\} \\
&\geqslant \frac{1}{p} \left(\frac{1}{\lambda + 1} \right).
\end{aligned}$$

This contradicts (2.1). Therefore, we have $|w(z)| < 1$ in E, and it follows that $\operatorname{Re}\{q(z)\} > 0$ in E. This completes our proof of Lemma 2.2. □

If we take $\lambda = 1$ in Lemma 2.2, then we have the following Lemma 2.3 by Nunokawa [5].

Lemma 2.3. *Let $q(z)$ be analytic in E with $q(0) = p$ and suppose that*

$$\operatorname{Re}\left\{\frac{zq'(z)}{[q(z)]^2}\right\} < \frac{1}{2p} \quad (z \in E).$$

Then $\operatorname{Re}\{q(z)\} > 0$ in E.

3. A Criterion for p-Valently Starlikeness

Theorem 3.1. *Let* $f(z) \in A(p),\ f(z) \neq 0$, *in* $0 < |z| < 1$ *and suppose that*

$$\text{(3.1)}\quad \operatorname{Re}\left\{ 1 + z\frac{\left[1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)\right]'}{\left[1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)\right]^2} \right\} < 1 + \frac{1}{k+1}\left(\frac{1}{2p}\right) \quad (z \in E).$$

Then $f(z) \in S(p,k)$.

Proof. Let us put

$$q(z) = \frac{1}{k+1}\left\{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right\} \quad (k > 0).$$

Then, $q(z)$ is analytic in E with $q(0) = p,\ q(z) \neq 0$ in E. We have

$$\frac{q'(z)}{q(z)} = \frac{\left(z\frac{f''(z)}{f'(z)}\right)' + \left(kz\frac{f'(z)}{f(z)}\right)'}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} = \frac{\frac{f''(z)}{f'(z)} + z\left(\frac{f''(z)}{f'(z)}\right)' + k\frac{f'(z)}{f(z)} + kz\left(\frac{f'(z)}{f(z)}\right)'}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}}.$$

Then, we obtain

$$z\frac{q'(z)}{q(z)} = \frac{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)} - 1}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} + z\frac{kz\left(\frac{f'(z)}{f(z)}\right)' + z\left(\frac{f''(z)}{f'(z)}\right)'}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}}$$

$$= 1 + \frac{z^2\left[\left(\frac{f''(z)}{f'(z)}\right)' + k\left(\frac{f'(z)}{f(z)}\right)'\right] - 1}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}},$$

or

$$
\begin{aligned}
(k+1)q(z) + z\frac{q'(z)}{q(z)} \\
&= 1 + \frac{z^2\left[\left(\frac{f''(z)}{f'(z)}\right)' + k\left(\frac{f'(z)}{f(z)}\right)'\right] - 1}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} + (k+1)q(z) \\
&= 1 + \frac{z^2\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)' + 2z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right) + z^2\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)^2}{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}} \\
&= 1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right) + z\frac{z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)' + \left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)}{\left(1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
1 + \frac{1}{k+1}z\frac{q'(z)}{[q(z)]^2} &= 1 + z\frac{z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)' + \left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)}{\left(1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right)^2} \\
&= 1 + z\frac{\left[1 + z\left(\frac{f''(z)}{f'(z)} + k\frac{f'(z)}{f(z)}\right)\right]'}{\left(1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right)^2}.
\end{aligned}
$$

From Lemma 2.3 and (3.1), we thus find that

$$
\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)} + kz\frac{f'(z)}{f(z)}\right\} \geqslant 0 \quad (z \in E, k > 0).
$$

This completes our proof of Theorem 3.1. □

If we take $\alpha = 0$, after writing $\frac{1}{k+1} = \alpha$ in (3.1), then we obtain M. Nunokawa's theorem as follows.

Theorem 3.2. *Let* $f(z) \in A(p)$, $f(z) \neq 0$, *in* $0 < |z| < 1$ *and suppose that*

$$\operatorname{Re}\left\{\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\} < 1 + \frac{1}{2p}, \quad z \in E.$$

Then $f(z) \in S(p)$.

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