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## A CRITERION FOR $p$ -VALENTLY STARLIKENESS

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## Abstract

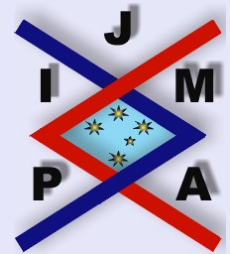
It is the purpose of the present paper to obtain some sufficient conditions for  $p$ -valently starlikeness for a certain class of functions which are analytic in the open unit disk  $E$ .

*2000 Mathematics Subject Classification:* 30C45, 31A05.

*Key words:*  $p$ -valently starlikeness, Jack Lemma.

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# 1. Introduction

Let  $A(p)$  be the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in  $E = \{z \in \mathbb{C} : |z| < 1\}$ .

A function  $f(z) \in A(p)$  is said to be  $p$ -valently starlike if and only if

$$\operatorname{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > 0 \quad (z \in E).$$

We denote by  $S(p)$  the subclass of  $A(p)$  consisting of functions which are  $p$ -valently in  $E$  (see, e.g., Goodman [1]).

Let

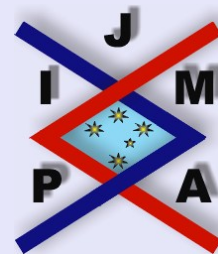
$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function  $f(z)$  of the form (1.1) is said to be  $\alpha$ -convex in  $E$  if it is regular,

$$\frac{f(z)}{z} f'(z) \neq 0,$$

and

$$(1.2) \quad \operatorname{Re} \left( \alpha \left( 1 + z \frac{f''(z)}{f'(z)} \right) + (1 - \alpha) z \frac{f'(z)}{f(z)} \right) \geq 0$$



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for all  $z$  in  $E$ . The set of all such functions is denoted by  $\alpha - CV$ , where  $\alpha$  is a real number. Of course, if  $\alpha = 1$ , then an  $\alpha$ -convex function is convex; and if  $\alpha = 0$ , an  $\alpha$ -convex function is starlike. Thus the sets  $\alpha - CV$  give a “continuous” passage from convex functions to starlike functions. Sakaguchi considers functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

where  $p$  is a positive integer, and he imposes the condition

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} + k z \frac{f'(z)}{f(z)} \right\} \geq 0$$

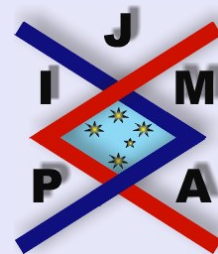
for  $z$  in  $E$ . He proved that if  $k = -1$ , there is only one function that satisfies (1.3), namely  $f(z) \equiv z^p$ . If  $-1 < k \leq 0$ , then  $f(z)$  is  $p$ -valent convex; and if  $0 < k$ , then  $f(z)$  is  $p$ -valent starlike. We can pass from (1.3) back to (1.2) if we divide by  $1 + k > 0$  and set  $\alpha = \frac{1}{1+k}$  [6]. We denote by  $S(p, k)$  the subclass  $A(p)$  consisting of functions which satisfy the condition (1.3).

Obradovic and Owa [7] have obtained a sufficient condition for starlikeness of  $f(z) \in A(1)$  which satisfies a certain condition for the modulus of

$$1 + \frac{z f''(z)}{f'(z)},$$

$$\frac{z f'(z)}{f(z)},$$

we recall their result as:



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**Theorem 1.1.** If  $f(z) \in A(1)$  satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < K \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in E),$$

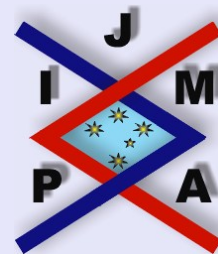
where  $K = 1.2849\dots$ , then  $f(z) \in S(1)$ .

Nunokawa [4] improved Theorem 1.1 by proving

**Theorem 1.2.** If  $f(z) \in A(p)$ , and if

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \left| \frac{zf'(z)}{f(z)} \right| \frac{1}{p} \log(4e^{p-1}) \quad (z \in E),$$

then  $f(z) \in S(p)$ .



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## 2. Preliminaries

In order to obtain our main result, we need the following lemma attributed to Jack [2] (given also by Miller and Mocanu [3]).

**Lemma 2.1.** *Let  $w(z)$  be analytic in  $E$  with  $w(0) = 0$ . If  $|w(z)|$  attains its maximum value in the circle  $|z| = r < 1$  at a point  $z_0$ , then we can write  $z_0 w'(z_0) = kw(z_0)$ , where  $k$  is a real number and  $k \geq 1$ .*

Making use of Lemma 2.1, we first prove

**Lemma 2.2.** *Let  $q(z)$  be analytic in  $E$  with  $q(0) = p$  and suppose that*

$$(2.1) \quad \operatorname{Re} \left\{ \frac{zq'(z)}{[q(z)]^2} \right\} < \frac{1}{p(\lambda + 1)} \quad (z \in E, 0 \leq \lambda \leq 1),$$

then  $\operatorname{Re}\{q(z)\} > 0$  in  $E$ .

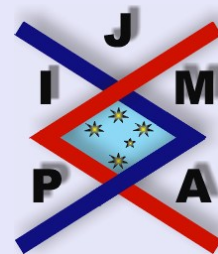
*Proof.* Let us put

$$q(z) = p \left\{ \left( \frac{1}{2} + \frac{1}{2}\lambda \right) \frac{1 + w(z)}{1 - w(z)} + \left( \frac{1}{2} - \frac{1}{2}\lambda \right) \frac{1 - w(z)}{1 + w(z)} \right\},$$

where  $0 \leq \lambda \leq 1$ .

Then  $w(z)$  is analytic in  $E$  with  $w(0) = 0$  and by an easy calculation, we have

$$1 + z \frac{q'(z)}{[q(z)]^2} = 1 + \frac{2}{p} \cdot \frac{(\lambda w^2(z) + 2w(z) + \lambda)zw'(z)}{(w^2(z) + 2\lambda w(z) + 1)^2}.$$



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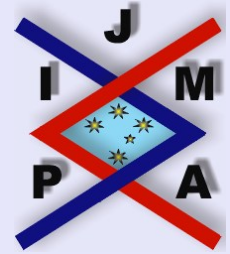
If we suppose that there exists a point  $z_0 \in E$  such that  $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$ , then, from Lemma 2.1, we have  $z_0 w'(z_0) = k w(z_0)$ , ( $k \geq 1$ ).

Putting  $w(z_0) = e^{i\theta}$ , we find that

$$\begin{aligned} z_0 \frac{q'(z_0)}{[q(z_0)]^2} &= \frac{2}{p} \cdot \frac{\lambda w^2(z_0) w'(z_0) z_0 + 2w(z_0) w'(z_0) z_0 + \lambda w'(z_0) z_0}{[w^2(z_0) + 2\lambda w(z_0) + 1]^2} \\ &= \frac{2k}{p} \cdot \frac{\lambda e^{3i\theta} + 2e^{2i\theta} + \lambda e^{i\theta}}{(e^{2i\theta} + 2\lambda e^{i\theta} + 1)^2} \\ &= \frac{2k}{p} \cdot \frac{(\lambda e^{3i\theta} + 2e^{2i\theta} + \lambda e^{i\theta})}{(e^{2i\theta} + 2\lambda e^{i\theta} + 1)^2} \cdot \frac{(e^{-2i\theta} + 2\lambda e^{-i\theta} + 1)^2}{(e^{-2i\theta} + 2\lambda e^{-i\theta} + 1)^2} \\ &= \frac{k}{p} \cdot \frac{\lambda \cos 3\theta + (4\lambda^2 + 2) \cos 2\theta + (11\lambda + 4\lambda^3) \cos \theta + (8\lambda^2 + 2)}{4(\lambda + \cos \theta)^4} \\ &= \frac{k}{p} \cdot \frac{(1 + \lambda \cos \theta)(\lambda + \cos \theta)^2}{(\lambda + \cos \theta)^4} \\ &= \frac{k}{p} \cdot \frac{1 + \lambda \cos \theta}{(\lambda + \cos \theta)^2}, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{Re} \left\{ z_0 \frac{q'(z_0)}{[q(z_0)]^2} \right\} &= \frac{k}{p} \cdot \frac{1 + \lambda \cos \theta}{(\lambda + \cos \theta)^2} = \frac{k}{p} \cdot \frac{\lambda^2 + \lambda \cos \theta + 1 - \lambda^2}{(\lambda + \cos \theta)^2} \\ &= \frac{k}{p} \left\{ \frac{\lambda}{(\lambda + \cos \theta)} + \frac{1 - \lambda^2}{(\lambda + \cos \theta)^2} \right\} \\ &\geq \frac{1}{p} \left( \frac{1}{\lambda + 1} \right). \end{aligned}$$



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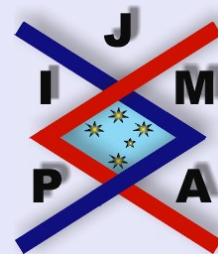
This contradicts (2.1). Therefore, we have  $|w(z)| < 1$  in  $E$ , and it follows that  $\operatorname{Re}\{q(z)\} > 0$  in  $E$ . This completes our proof of Lemma 2.2.  $\square$

If we take  $\lambda = 1$  in Lemma 2.2, then we have the following Lemma 2.3 by Nunokawa [5].

**Lemma 2.3.** *Let  $q(z)$  be analytic in  $E$  with  $q(0) = p$  and suppose that*

$$\operatorname{Re} \left\{ \frac{zq'(z)}{[q(z)]^2} \right\} < \frac{1}{2p} \quad (z \in E).$$

*Then  $\operatorname{Re}\{q(z)\} > 0$  in  $E$ .*




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### 3. A Criterion for $p$ -Valently Starlikeness

**Theorem 3.1.** Let  $f(z) \in A(p)$ ,  $f(z) \neq 0$ , in  $0 < |z| < 1$  and suppose that

$$(3.1) \operatorname{Re} \left\{ 1 + z \frac{\left[ 1 + z \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right) \right]'}{\left[ 1 + z \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right) \right]^2} \right\} < 1 + \frac{1}{k+1} \left( \frac{1}{2p} \right) \quad (z \in E).$$

Then  $f(z) \in S(p, k)$ .

*Proof.* Let us put

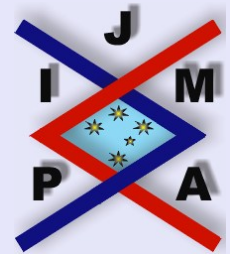
$$q(z) = \frac{1}{k+1} \left\{ 1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)} \right\} \quad (k > 0).$$

Then,  $q(z)$  is analytic in  $E$  with  $q(0) = p$ ,  $q(z) \neq 0$  in  $E$ . We have

$$\frac{q'(z)}{q(z)} = \frac{\left( z \frac{f''(z)}{f'(z)} \right)' + \left( kz \frac{f'(z)}{f(z)} \right)' }{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}} = \frac{\frac{f''(z)}{f'(z)} + z \left( \frac{f''(z)}{f'(z)} \right)' + k \frac{f'(z)}{f(z)} + kz \left( \frac{f'(z)}{f(z)} \right)'}{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}}.$$

Then, we obtain

$$\begin{aligned} z \frac{q'(z)}{q(z)} &= \frac{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)} - 1}{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}} + z \frac{kz \left( \frac{f'(z)}{f(z)} \right)' + z \left( \frac{f''(z)}{f'(z)} \right)'}{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}} \\ &= 1 + \frac{z^2 \left[ \left( \frac{f''(z)}{f'(z)} \right)' + k \left( \frac{f'(z)}{f(z)} \right)' \right] - 1}{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}}, \end{aligned}$$



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or

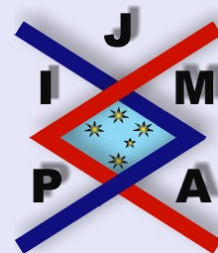
$$\begin{aligned}
 & (k+1)q(z) + z \frac{q'(z)}{q(z)} \\
 &= 1 + \frac{z^2 \left[ \left( \frac{f''(z)}{f'(z)} \right)' + k \left( \frac{f'(z)}{f(z)} \right)' \right] - 1}{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}} + (k+1)q(z) \\
 &= 1 + \frac{z^2 \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right)' + 2z \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right) + z^2 \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right)^2}{1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)}} \\
 &= 1 + z \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right) + z \frac{\left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right)' + \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right)}{\left( 1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)} \right)}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 1 + \frac{1}{k+1} z \frac{q'(z)}{[q(z)]^2} &= 1 + z \frac{\left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right)' + \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right)}{\left( 1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)} \right)^2} \\
 &= 1 + z \frac{\left[ 1 + z \left( \frac{f''(z)}{f'(z)} + k \frac{f'(z)}{f(z)} \right) \right]'}{\left( 1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)} \right)^2}.
 \end{aligned}$$

From Lemma 2.3 and (3.1), we thus find that

$$\operatorname{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} + kz \frac{f'(z)}{f(z)} \right\} \geq 0 \quad (z \in E, k > 0).$$



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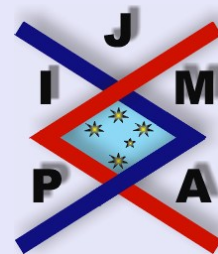
This completes our proof of Theorem 3.1. □

If we take  $\alpha = 0$ , after writing  $\frac{1}{k+1} = \alpha$  in (3.1), then we obtain M. Nunokawa's theorem as follows.

**Theorem 3.2.** *Let  $f(z) \in A(p)$ ,  $f(z) \neq 0$ , in  $0 < |z| < 1$  and suppose that*

$$\operatorname{Re} \left\{ \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \right\} < 1 + \frac{1}{2p}, \quad z \in E.$$

*Then  $f(z) \in S(p)$ .*



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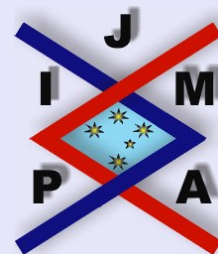
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