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MATRIX EQUALITIES AND INEQUALITIES INVOLVING KHATRI-RAO AND TRACY-SINGH SUMS

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ABSTRACT. The Khatri-Rao and Tracy-Singh products for partitioned matrices are viewed as generalized Hadamard and generalized Kronecker products, respectively. We define the Khatri-Rao and Tracy-Singh sums for partitioned matrices as generalized Hadamard and generalized Kronecker sums and derive some results including matrix equalities and inequalities involving the two sums. Based on the connection between the Khatri-Rao and Tracy-Singh products (sums) and use mainly Liu's, Mond and Pečarić's methods to establish new inequalities involving the Khatri-Rao product (sum). The results lead to inequalities involving Hadamard and Kronecker products (sums), as a special case.

Key words and phrases: Kronecker product (sum), Hadamard product (sum), Khatri-Rao product (sum), Tracy-Singh product (sum), Positive (semi)definite matrix, Unitarily invariant norm, Spectral norm, P-norm, Moore-Penrose inverse.

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1. INTRODUCTION

The Hadamard and Kronecker products are studied and applied widely in matrix theory, statistics, econometrics and many other subjects. Partitioned matrices are often encountered in statistical applications.

For partitioned matrices, The Khatri-Rao product viewed as a generalized Hadamard product, is discussed and used in [7, 6, 14] and the Tracy-Singh product, as a generalized Kronecker product, is discussed and applied in [7, 5, 12]. Most results provided are equalities associated with the products. Rao, Kleffe and Liu in [13, 8] presented several matrix inequalities involving the Khatri-Rao product, which seem to be most existing results. In [7], Liu established the connection between Khatri-Rao and Tracy-Singh products based on two selection matrices Z_1 and Z_2. This connection play an important role to give inequalities involving the two products

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with statistical applications. In [10], Mond and Pečarić presented matrix versions, with matrix weights. In [2, (2004)], Hiai and Zhan proved the following inequalities:

$$(*)\qquad \frac{\|AB\|}{\|A\|\cdot\|B\|} \le \frac{\|A+B\|}{\|A\|+\|B\|}, \qquad \frac{\|A\circ B\|}{\|A\|\cdot\|B\|} \le \frac{\|A+B\|}{\|A\|+\|B\|}$$

for any invariant norm with $\|\mathrm{diag}(1,0,\dots,0)\| \ge 1$ and A, B are nonzero positive definite matrices.

In the present paper, we make a further study of the Khatri-Rao and Tracy-Singh products. We define the Khatri-Rao and Tracy-Singh sums for partitioned matrices and use mainly Liu's, Mond and Pečarić's methods to obtain new inequalities involving these products (sums).We collect several known inequalities which are derived as a special cases of some results obtained. We generalize the inequalities in Eq (*) involving the Hadamard product (sum) and the Kronecker product (sum).

2. Basic Definitions and Results

2.1. Basic Definitions on Matrix Products.

We introduce the definitions of five known matrix products for non-partitioned and partitioned matrices. These matrix products are defined as follows:

Definition 2.1. Consider matrices $A = (a_{ij})$ and $C = (c_{ij})$ of order $m \times n$ and $B = (b_{kl})$ of order $p \times q$. The Kronecker and Hadamard products are defined as follows:

(1) *Kronecker product:*

$$A \otimes B = (a_{ij}B)_{ij}\,, \tag{2.1}$$

where $a_{ij}B$ is the *ij*$^{\text{th}}$ submatrix of order $p \times q$ and $A \otimes B$ of order $mp \times nq$.

(2) *Hadamard product:*

$$A \circ C = (a_{ij}c_{ij})_{ij}\,, \tag{2.2}$$

where $a_{ij}c_{ij}$ is the *ij*$^{\text{th}}$ scalar element and $A \circ C$ is of order $m \times n$.

Definition 2.2. Consider matrices $A = (a_{ij})$ and $B = (b_{kl})$ of order $m \times m$ and $n \times n$ respectively. The *Kronecker sum* is defined as follows:

$$A \oplus B = A \otimes I_n + I_m \otimes B, \tag{2.3}$$

where I_n and I_m are identity matrices of order $n \times n$ and $m \times m$ respectively, and $A \oplus B$ of order $mn \times mn$.

Definition 2.3. Consider matrices A and C of order $m\times n$, and B of order $p\times q$. Let $A = (A_{ij})$ be partitioned with A_{ij} of order $m_i \times n_j$ as the *ij*$^{\text{th}}$ submatrix,$C = (C_{ij})$ be partitioned with C_{ij} of order $m_i \times n_j$ as the *ij*$^{\text{th}}$ submatrix, and $B = (B_{kl})$ be partitioned with B_{kl} of order $p_k \times q_l$ as the *kl*$^{\text{th}}$ submatrix, where,$m = \sum_{i=1}^{r} m_i$, $n = \sum_{j=1}^{s} n_j$, $p = \sum_{k=1}^{t} p_k$, $q = \sum_{l=1}^{h} q_l$ are partitions of positive integers m, n, p, and q. The Tracy-Singh and Khatri-Rao products are defined as follows:

(1) *Tracy-Singh product:*

$$A\Pi B = (A_{ij}\Pi B)_{ij} = \left((A_{ij} \otimes B_{kl})_{kl}\right)_{ij}\,, \tag{2.4}$$

where A_{ij} is the *ij*$^{\text{th}}$ submatrix of order $m_i \times n_j$, B_{kl} is the *kl*$^{\text{th}}$ submatrix of order $p_k \times q_l$, $A_{ij}\Pi B$ is the *ij*$^{\text{th}}$ submatrix of order $m_i p \times n_j q$, $A_{ij} \otimes B_{kl}$ is the *kl*$^{\text{th}}$ submatrix of order $m_i p_k \times n_j q_l$ and $A\Pi B$ of order $mp \times nq$.

Note that

(i) For a non partitioned matrix A, their $A\Pi B$ is $A \otimes B$, i.e., for $A = (a_{ij})$, where a_{ij} is scalar, we have,

$$\begin{aligned} A\Pi B &= (a_{ij}\Pi B)_{ij} \\ &= \left((a_{ij} \otimes B_{kl})_{kl}\right)_{ij} \\ &= \left((a_{ij}B_{kl})_{kl}\right)_{ij} = (a_{ij}B)_{ij} = A \otimes B. \end{aligned}$$

(ii) For column wise partitioned A and B, their $A\Pi B$ is $A \otimes B$.

(2) *Khatri-Rao product:*

$$A * B = (A_{ij} \otimes B_{ij})_{ij}\,, \tag{2.5}$$

where A_{ij} is the *ij*$^{\text{th}}$ submatrix of order $m_i \times n_j$, B_{ij} is the *ij*$^{\text{th}}$ submatrix of order $p_i \times q_j$, $A_{ij} \otimes B_{ij}$ is the *ij*$^{\text{th}}$ submatrix of order $m_i p_i \times n_j q_j$ and $A * B$ of order $M \times N$ $\left(M = \sum_{i=1}^{r} m_i p_i,\ N = \sum_{j=1}^{s} n_j q_j\right)$.

Note that

(i) For a non partitioned matrix A, their $A * B$ is $A \otimes B$, i.e., for $A = (a_{ij})$,where a_{ij} is scalar, we have,

$$A * B = (a_{ij} \otimes B_{ij})_{ij} = (a_{ij}B)_{ij} = A \otimes B.$$

(ii) For non partitioned matrices A and B, their $A * B$ is $A \circ B$, i.e., for $A = (a_{ij})$ and $B = (b_{ij})$, where a_{ij} and b_{ij} are scalars, we have,

$$A * B = (a_{ij} \otimes b_{ij})_{ij} = (a_{ij}b_{ij})_{ij} = A \circ B.$$

2.2. **Basic Connections and Results on Matrix Products.** We introduce the connection between the Katri-Rao and Tracy-Singh products and the connection between the Kronecker and Hadamard products, as a special case, which are important in creating inequalities involving these products. We write $A \geq B$ in the Löwner ordering sense that $A - B \geq 0$ is positive semi-definite, for symmetric matrices A and B of the same order and A^+ and A^* indicate the Moore-Penrose inverse and the conjugate of the matrix A, respectively.

Lemma 2.1. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two scalar matrices of order $m \times n$. Then (see* [15]*)*

$$A \circ B = K_1'(A \otimes B)K_2 \tag{2.6}$$

where K_1 and K_2 are two selection matrices of order $n^2 \times n$ and $m^2 \times m$, respectively, such that $K_1'K_1 = I_m$ and $K_2'K_2 = I_n$.

In particular, for $m = n$, we have $K_1 = K_2 = K$ and

$$A \circ B = K'(A \otimes B)K \tag{2.7}$$

Lemma 2.2. *Let A and B be compatibly partitioned. Then (see* [8, p. 177-178] *and* [7, p. 272]*)*

$$A * B = Z_1'\left(A\Pi B\right) Z_2, \tag{2.8}$$

where Z_1 and Z_2 are two selection matrices of zeros and ones such that $Z_1'Z_1 = I_1$ and $Z_2'Z_2 = I_2$, where I_1 and I_2 are identity matrices.

In particular, when A and B are square compatibly partitioned matrices, then we have $Z_1 = Z_2 = Z$ such that $Z'Z = I$ and

$$A * B = Z' \left(A \Pi B \right) Z. \tag{2.9}$$

Note that, for non-partitioned matrices A, B, Z_1 and Z_2, Lemma 2.2 leads to Lemma 2.1, as a special case.

Lemma 2.3. *Let A, B, C, D and F be compatibly partitioned matrices. Then*

$$(A\Pi B)(C\Pi D) = (AC)\Pi(BD) \tag{2.10}$$

$$(A\Pi B)^+ = A^+\Pi B^+ \tag{2.11}$$

$$(A + C)\Pi(B + D) = A\Pi B + A\Pi D + C\Pi B + C\Pi D \tag{2.12}$$

$$(A\Pi B)^* = A^*\Pi B^* \tag{2.13}$$

$$A\Pi B \neq B\Pi A \qquad \textit{in general} \tag{2.14}$$

$$A * B \neq B * A \qquad \textit{in general} \tag{2.15}$$

$$B * F = F * B \quad \textit{where} \quad F = (f_{ij}) \quad \textit{and} \quad f_{ij} \quad \textit{is a scalar} \tag{2.16}$$

$$(A * B)^* = A^* * B^* \tag{2.17}$$

$$(A + C) * (B + D) = A * B + A * D + C * B + C * D \tag{2.18}$$

$$(A * B)\Pi(C * D) = (A\Pi C) * (B\Pi D) \tag{2.19}$$

Proof. Straightforward. □

Lemma 2.4. *Let A and B be compatibly partitioned matrices. Then*

$$(A\Pi B)^r = A^r\Pi B^r, \tag{2.20}$$

for any positive integer r.

Proof. The proof is by induction on r and using Eq. (2.10). □

Theorem 2.5. *Let $A \geq 0$ and $B \geq 0$ be compatibly partitioned matrices. Then*

$$(A\Pi B)^\alpha = A^\alpha\Pi B^\alpha \tag{2.21}$$

for any positive real α.

Proof. By using Eq (2.20), we have $A\Pi B = (A^{1/n}\Pi B^{1/n})^n$, for any positive integer n. So it follows that $(A\Pi B)^{1/n} = A^{1/n}\Pi B^{1/n}$. Now $(A\Pi B)^{m/n} = A^{m/n}\Pi B^{m/n}$, for any positive integers n, m. The Eq (2.21) now follows by a continuity argument. □

Corollary 2.6. *Let A and B be compatibly partitioned matrices. Then*

$$|A\Pi B| = |A| \, \Pi \, |B| \, , \quad \textit{where} \quad |A| = (A^*A)^{1/2} \tag{2.22}$$

Proof. Applying Eq (2.10) and Eq (2.21), we get the result. □

Theorem 2.7. *Let $A = (A_{ij})$ and $B = (B_{kl})$ be partitioned matrices of order $m \times m$, and $n \times n$ respectively, where $m = \sum_{i=1}^{r} m_i$, $n = \sum_{k=1}^{t} n_k$ .Then*

$$\text{(a)} \quad \operatorname{tr}(A\Pi B) = \operatorname{tr}(A) \cdot \operatorname{tr}(B) \tag{2.23}$$

$$\text{(b)} \quad \|A\Pi B\|_p = \|A\|_p \, \|B\|_p \, , \quad \textit{where} \quad \|A\|_p = \left[\operatorname{tr} |A|^p\right]^{1/p} , \ \textit{for all } 1 \leq p < \infty. \tag{2.24}$$

Proof. (a) Straightforward.
(b) Applying Eq (2.22) and Eq (2.23), we get the result. □

Theorem 2.8. *Let A, B and I be compatibly partitioned matrices. Then*

$$(A\Pi I)(I\Pi B) = (I\Pi B)(A\Pi I) = A\Pi B. \tag{2.25}$$

If $f(A)$ is an analytic function on a region containing the eigenvalues of A, then

$$f(I\Pi A) = I\Pi f(A) \quad \text{and} \quad f(A\Pi I) = f(A)\Pi I \tag{2.26}$$

Proof. The proof of Equation (2.25) is straightforward on applying Eq (2.10).

Equation (2.26) can be proved as follows:

Since $f(A)$is an analytic function, then $f(A) = \sum_{k=0}^{\infty} \alpha_k A^k$. Applying Eq (2.10) we get:

$$f(I\Pi A) = \sum_{k=0}^{\infty} \alpha_k (I\Pi A)^k = \sum_{k=0}^{\infty} \alpha_k (I\Pi A^k) = I\Pi \sum_{k=0}^{\infty} \alpha_k A^k = I\Pi f(A).$$

□

Corollary 2.9. *Let A, B and I be compatibly partitioned matrices. Then*

$$e^{A\Pi I} = e^A \Pi I \quad \text{and} \quad e^{I\Pi A} = I\Pi e^A. \tag{2.27}$$

Lemma 2.10. *Let $H \geq 0$ be a $n \times n$ matrix with nonzero eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k$ $(k \leq n)$ and X be a $m \times m$ matrix such that $X = H^0 X$, where $H^0 = HH^+$. Then (see* [6, *Section 2.3*]*)*

$$(X'HX)^+ \leq X^+ H^+ X'^+ \leq \frac{(\lambda_1 + \lambda_k)^2}{(4\lambda_1 \lambda_k)} (X'HX)^+. \tag{2.28}$$

Theorem 2.11. *Let $A \geq 0$ and $B \geq 0$ be compatibly partitioned matrices such that $A^0 = AA^+$ and $B^0 = BB^+$. Then (see* [8, Section 3]*)*

$$(A * B^0 + A^0 * B)(A * B)^+(A * B^0 + A^0 * B) \leq A * B^+ + A^+ * B + 2A^0 * B^0 \tag{2.29}$$

Theorem 2.12. *Let $A > 0$ and $B > 0$ be $n\times n$ compatibly partitioned matrices with eigenvalues contained in the interval between m and M $(M \geq m)$. Let I be a compatible identity matrix. Then (see* [8, Section 3]*).*

$$A * B^{-1} + A^{-1} * B \leq \frac{m^2 + M^2}{mM} I \quad \text{and} \quad A * A^{-1} \leq \frac{m^2 + M^2}{2mM} I \tag{2.30}$$

3. MAIN RESULTS

3.1. On the Tracy-Singh Sum.

Definition 3.1. Consider matrices A and B of order $m\times m$ and $n\times n$ respectively. Let $A = (A_{ij})$ be partitioned with A_{ij} of order $m_i \times m_i$ as the ij$^{\text{th}}$ submatrix, and let $B = (B_{ij})$ be partitioned with B_{ij} of order $n_k \times n_k$ as the ij$^{\text{th}}$ submatrix $\left(m = \sum_{i=1}^{r} m_i,\ n = \sum_{k=1}^{t} n_k\right)$.

The Tracy-Singh sum is defined as follows:

$$A\nabla B = A\Pi I_n + I_m \Pi B, \tag{3.1}$$

where $I_n = I_{n_1+n_2+\cdots+n_t} = \text{blockdiag}(I_{n_1}, I_{n_2}, \ldots, I_{n_t})$ is an $n \times n$ identity matrix, $I_m = I_{m_1+m_2+\cdots+m_r} = \text{blockdiag}(I_{m_1}, I_{m_2}, \ldots, I_{m_r})$ is an $m \times m$ identity matrix, I_{n_k} is an $n_k \times n_k$ identity matrix $(k = 1, \ldots, t)$, I_{m_i} is an $m_i \times m_i$ identity matrix $(i = 1, \ldots, r)$ and $A\nabla B$ is of order $mn \times mn$.

Note that for non-partitioned matrices A and B, their $A\nabla B$ is $A \oplus B$.

Theorem 3.1. *Let $A \geq 0$, $B \geq 0$, $C \geq 0$ and $D \geq 0$ be compatibly partitioned matrices. Then*

$$(A\nabla B)(C\nabla D) \geq AC\nabla BD. \tag{3.2}$$

Proof. Applying Eq (3.1) and Eq (2.10), we have

$$\begin{aligned}(A\nabla B)(C\nabla D) &= (A\Pi I + I\Pi B)(C\Pi I + I\Pi D)\\ &= (A\Pi I)(C\Pi I) + (A\Pi I)(I\Pi D) + (I\Pi B)(C\Pi I) + (I\Pi B)(I\Pi D)\\ &= AC\Pi I + A\Pi D + C\Pi B + I\Pi BD\\ &= AC\nabla BD + A\Pi D + C\Pi B \geq AC\nabla BD.\end{aligned}$$

In special cases of Eq (3.2), if $C = A^*$, $D = B^*$, we have

$$(A\nabla B)(A\nabla B)^* \geq AA^*\nabla BB^* \tag{3.3}$$

and if $C = A$, $D = B$, we have

$$(A\nabla B)^2 \geq A^2\nabla B^2. \tag{3.4}$$

More generally, it is easy by induction on w we can show that if$A \geq 0$ and $B \geq 0$ are compatibly partitioned matrices. Then

$$(A\nabla B)^w = A^w\nabla B^w + \sum_{k=1}^{w-1}\binom{w}{k}(A^{w-k}\Pi B^k); \tag{3.5}$$

$$(A\nabla B)^w \geq A^w\nabla B^w \tag{3.6}$$

for any positive integer w. □

Theorem 3.2. *Let A and B be partitioned matrices of order $m \times m$ and $n \times n$, respectively, $\left(m = \sum_{i=1}^{r} m_i,\ n = \sum_{k=1}^{t} n_k\right)$. Then*

$$\operatorname{tr}(A\nabla B) = n \cdot \operatorname{tr}(A) + m \cdot \operatorname{tr}(B), \tag{3.7}$$

$$\|A\nabla B\|_p \leq \sqrt[p]{n}\,\|A\|_p + \sqrt[p]{m}\,\|B\|_p\,, \tag{3.8}$$

where $\|A\|_p = [\operatorname{tr}|A|^p]^{1/p}$, $1 \leq p < \infty$, and

$$e^{A\nabla B} = e^A\Pi e^B. \tag{3.9}$$

Proof. For the first part, on applying Eq (2.23), we obtain

$$\begin{aligned}\operatorname{tr}(A\nabla B) &= \operatorname{tr}\left[(A\Pi I_n) + (I_m\Pi B)\right]\\ &= \operatorname{tr}(A\Pi I_n) + \operatorname{tr}(I_m\Pi B)\\ &= \operatorname{tr}(A)\operatorname{tr}(I_n) + \operatorname{tr}(I_m)\operatorname{tr}(B)\\ &= n\cdot\operatorname{tr}(A) + m\cdot\operatorname{tr}(B).\end{aligned}$$

To prove (3.8), we apply Eq (2.24), to get

$$\begin{aligned}\|A\nabla B\|_p &= \|(A\Pi I_n) + (I_m\Pi B)\|_p\\ &\leq \|A\Pi I_n\|_p + \|I_m\Pi B\|_p\\ &= \|A\|_p\,\|I_n\|_p + \|I_m\|_p\,\|B\|_p\\ &= \sqrt[p]{n}\,\|A\|_p + \sqrt[p]{m}\,\|B\|_p\,.\end{aligned}$$

For the last part, applying Eq (2.25), Eq (2.27) and Eq (2.10), we have

$$\begin{aligned}e^{A\nabla B} &= e^{(A\Pi I_n)+(I_m\Pi B)}\\ &= e^{(A\Pi I_n)}e^{(I_m\Pi B)}\\ &= (e^A\Pi I_n)(I_m\Pi e^B) = e^A\Pi e^B.\end{aligned}$$

□

Theorem 3.3. *Let A and Bbe non singular partitioned matrices of order $m \times m$ and $n \times n$ respectively, ($m = \sum_{i=1}^{r} m_i$, $n = \sum_{k=1}^{t} n_k$).Then*

$$(A\nabla B)^{-1} = (A^{-1}\nabla B^{-1})^{-1}(A^{-1}\Pi B^{-1}) \tag{3.10 (i)}$$

$$(A\nabla B)^{-1} = (A^{-1}\Pi I_n)(A^{-1}\nabla B^{-1})^{-1}(I_m\Pi B^{-1}) \tag{3.11 (ii)}$$

$$(A\nabla B)^{-1} = (I_m\Pi B^{-1})(A^{-1}\nabla B^{-1})^{-1}(A^{-1}\Pi I_n) \tag{3.12 (iii)}$$

Proof. (i) Applying Eq (2.10), we have

$$\begin{aligned}
(A\nabla B)^{-1} &= [I_m\Pi B + A\Pi I_n]^{-1} \\
&= [(I_m\Pi B)(I_m\Pi I_n) + (I_m\Pi B)(A\Pi B^{-1})]^{-1} \\
&= [(I_m\Pi B)(I_m\Pi I_n + A\Pi B^{-1})]^{-1} \\
&= [(I_m\Pi I_n + A\Pi B^{-1})]^{-1}[I_m\Pi B]^{-1} \\
&= [(A\Pi I_n)(A^{-1}\Pi I_n) + (A\Pi I_n)(I_m\Pi B^{-1})]^{-1}[I_m\Pi B^{-1}] \\
&= [(A\Pi I_n)\{A^{-1}\Pi I_n + I_m\Pi B^{-1}\}]^{-1}[I_m\Pi B^{-1}] \\
&= [(A\Pi I_n)(A^{-1}\nabla B^{-1})]^{-1}[I_m\Pi B^{-1}] \\
&= (A^{-1}\nabla B^{-1})^{-1}(A^{-1}\Pi I_n)(I_m\Pi B^{-1}) \\
&= (A^{-1}\nabla B^{-1})^{-1}(A^{-1}\Pi B^{-1}).
\end{aligned}$$

Similarly, we obtain (ii) and (iii). □

Theorem 3.4. *Let $A \geq 0$ and I be compatibly partitioned matrices such that $A^{+}\Pi I = I\Pi A^{+}$. Then*

$$A\nabla A^{+} \geq 2AA^{+}\Pi I. \tag{3.13}$$

Proof. We know that $A\nabla I = A\Pi I + I\Pi I > A\Pi I$. Denote $H = M\Pi I \geq 0$. By virtue of $H + H^{+} \geq 2HH^{+}$ and Eq (2.10), we have

$$A\Pi I + (A\Pi I)^{+} \geq 2(A\Pi I)(A\Pi I)^{+} = 2AA^{+}\Pi I$$

Since, $A^{+}\Pi I = I\Pi A^{+}$, we get the result. □

3.2. **On the Khatri-Rao Sum.**

Definition 3.2. Let A, B, I_n and I_m be partitioned as in Definition 3.1. Then the *Khatri-Rao sum* is defined as follows:

$$A\infty B = A * I_n + I_m * B \tag{3.14}$$

Note that, for non-partitioned matrices A and B, their $A\infty B$ is $A \oplus B$, and for non-partitioned matrices A, B, I_n and I_m, their $A\infty B$ is $A \bullet B$ (*Hadamard sum, see Definition 4.1, Eq(4.1), Section 4*).

Theorem 3.5. *Let A and B be compatibly partitioned matrices. Then*

$$A\infty B = Z'(A\nabla B)Z, \tag{3.15}$$

where Z is a selection matrix as in Lemma 2.2.

Proof. Applying Eq (2.9), we have $A * I = Z'(A\Pi I)Z$, $I * B = Z'(I\Pi B)Z$ and

$$A\infty B = A * I + I * B = Z'\left(A\Pi I\right)Z + Z'\left(I\Pi B\right)Z = Z'(A\nabla B)Z.$$

□

Corollary 3.6. *Let $A \geq 0$ and I be compatibly partitioned matrices such that $A^{+}\Pi I = I\Pi A^{+}$. Then*

$$A\infty A^{+} \geq 2AA^{+} * I \tag{3.16}$$

Proof. Applying Eq(3.13) and Eq (3.15), we get the result. □

Corollary 3.7. *Let $A > 0$ be compatibly partitioned with eigenvalues contained in the interval between m and M $(M \geq m)$. Let I be a compatible identity matrix such that $A^{-1}\infty I = I\infty A^{-1}$. Then*

$$A\infty A^{-1} \leq \frac{m^2 + M^2}{mM} I. \tag{3.17}$$

Proof. Applying Eq (2.30) and taking $B = I$, we get the result. □

Corollary 3.8. *Let $A \geq 0$ and I be compatibly partitioned, where $A^0 = AA^{+}$ such that $A^0 * I = I * A^0$. Then*

$$(A\infty A^0)(A * I)^{+}(A\infty A^0) \leq A * I + A^{+} * I + 2A^0 * I \tag{3.18}$$

and if $A^{+} * I = I * A^{+}$, we have

$$(A\infty A^0)(A * I)^{+}(A\infty A^0) \leq A\infty A^{+} + 2A^0 * I. \tag{3.19}$$

Proof. Applying Eq (2.29) and taking $B = I$, we get the results. □

Mond and Pečarić (see [10]) proved the following result:

If X_j $(j = 1, 2, \ldots, k)$ are positive definite Hermitian matrices of order $n\times n$ with eigenvalues in the interval $[m, M]$ and U_j $(j = 1, 2, \ldots, k)$ are $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then

(a) For $p < 0$ or $p > 1$, we have

$$\sum_{j=1}^{k} U_j X_j^p U_j^* \leq \lambda \left(\sum_{j=1}^{k} U_j X_j U_j^*\right)^p \tag{3.20}$$

where,

$$\lambda = \frac{\gamma^p - \gamma}{(p-1)(\gamma - 1)} \left\{\frac{p(\gamma - \gamma^p)}{(1-p)(\gamma^p - 1)}\right\}^{-p}, \qquad \gamma = \frac{M}{m}. \tag{3.21}$$

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.20).

(b) For $p < 0$ or $p > 1$, we have

$$\left(\sum_{j=1}^{k} U_j X_j^p U_j^*\right) - \left(\sum_{j=1}^{k} U_j X_j U_j^*\right)^p \leq \alpha I, \tag{3.22}$$

where,

$$\alpha = m^p - \left\{\frac{M^p - m^p}{p(M-m)}\right\}^{\frac{p}{p-1}} + \frac{M^p - m^p}{(M-m)} \left[\left\{\frac{M^p - m^p}{p(M-m)}\right\}^{\frac{1}{p-1}} - m\right]. \tag{3.23}$$

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.22).

We have an application to the Khatri-Rao product and Khatri-Rao sum.

Theorem 3.9. *Let A and B be positive definite Hermitian compatibly partitioned matrices and let m and M be, respectively, the smallest and the largest eigenvalues of $A\Pi B$. Then*

(a) *For p a nonzero integer, we have*

$$A^p * B^p \leq \lambda (A * B)^p \tag{3.24}$$

where, λ is given by Eq (3.21).

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.24).

(b) *For p a nonzero integer, we have*

$$(A^p * B^p) - (A * B)^p \leq \alpha I, \tag{3.25}$$

where α is given by Eq (3.23).

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.25).

Proof. In Eq (3.20) and Eq (3.22), take $k = 1$ and instead of U^*, use Z, the selection matrix which satisfy the following property:

$$A * B = Z'(A \Pi B)Z, \quad Z'Z = I.$$

Making use of the fact in Eq (2.21) that for any real n (positive or negative), we have

$$(A \Pi B)^n = A^n \Pi B^n,$$

then, with Z', $A \Pi B$, Z substituted for U, X, U^*, we have from Eq (3.20)

$$\begin{aligned} A^p * B^p &= Z'(A^p * B^p)Z \\ &= Z'(A * B)^p Z \\ &\leq \lambda \left\{ Z'(A \Pi B)Z \right\}^p = \lambda (A * B)^p, \end{aligned}$$

where, λ is given by Eq (3.21)

Similarly, from Eq (3.22), we obtain for

$$(A^p * B^p) - (A * B)^p \leq \alpha I$$

where, α is given by Eq (3.23).

Special cases include from Eq (3.24):

(2.1) For $p = 2$, we have

$$A^2 * B^2 \leq \frac{(M+m)^2}{4Mm} \left\{ A * B \right\}^2 \tag{3.26}$$

(2.2) For $p = -1$, we have

$$A^{-1} * B^{-1} \leq \frac{(M+m)^2}{4Mm} \left\{ A * B \right\}^{-1} \tag{3.27}$$

Similarly, special cases include from Eq (3.25):

(2.1) For $p = 2$, we have

$$(A^2 * B^2) - (A * B)^2 \leq \frac{1}{4}(M - m)^2 I \tag{3.28}$$

(2.2) For $p = -1$, we have

$$(A^{-1} * B^{-1}) - (A * B)^{-1} \leq \frac{\sqrt{M} - \sqrt{m}}{Mm} \left\{ I \right\}, \tag{3.29}$$

where results in Eq (3.26), Eq (3.27), and Eq (3.28) are given in [7]. □

Theorem 3.10. *Let A and B be positive definite Hermitian compatibly partitioned matrices. Let m_1 and M_1 be, respectively, the smallest and the largest eigenvalues of $A \Pi I$ and m_2 and M_2, respectively, the smallest and the largest eigenvalues of $I \Pi B$. Then*

(a) *For p a nonzero integer, we have*

$$A^p \infty B^p \leq \max\{\lambda_1, \lambda_2\}(A\infty B)^p \tag{3.30}$$

where,

$$\lambda_1 = \frac{(\gamma_1^p - \gamma_1)}{[(p-1)(\gamma_1 - 1)]}\left\{\frac{p(\gamma_1 - \gamma_1^p)}{[(1-p)(\gamma_1^p - 1)]}\right\}^{-p}, \qquad \gamma_1 = \frac{M_1}{m_1}, \tag{3.31}$$

$$\lambda_2 = \frac{(\gamma_2^p - \gamma_2)}{[(p-1)(\gamma_2 - 1)]}\left\{\frac{p(\gamma_2 - \gamma_2^p)}{[(1-p)(\gamma_2^p - 1)]}\right\}^{-p}, \quad \gamma_2 = \frac{M_2}{m_2}. \tag{3.32}$$

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.30).

(b) *For p a nonzero integer, we have*

$$(A^p \infty B^p) - (A\infty B)^p \leq \max\{\alpha_1, \alpha_2\} I \tag{3.33}$$

where,

$$\alpha_1 = m_1^p - \left\{\frac{M_1^p - m_1^p}{p(M_1 - m_1)}\right\}^{\frac{p}{p-1}} + \frac{M_1^p - m_1^p}{M_1 - m_1}\left\{\left\{\frac{M_1^p - m_1^p}{p(M_1 - m_1)}\right\}^{\frac{1}{p-1}} - m_1\right\} \tag{3.34}$$

$$\alpha_2 = m_2^p - \left\{\frac{M_2^p - m_2^p}{p(M_2 - m_2)}\right\}^{\frac{p}{p-1}} + \frac{M_2^p - m_2^p}{M_2 - m_2}\left\{\left\{\frac{M_2^p - m_2^p}{p(M_2 - m_2)}\right\}^{\frac{1}{p-1}} - m_2\right\} \tag{3.35}$$

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.33).

Proof. Applying Eq (3.24), we have

$$A^p * I = A^p * I^p \leq \lambda_1 (A * I)^p$$

$$I * B^p = I^p * B^p \leq \lambda_2 (I * B)^p$$

Now,

$$\begin{aligned} A^p \infty B^p &= A^p * I + I * B^p \\ &\leq \lambda_1 (A * I)^p + \lambda_2 (I * B)^p \\ &\leq \max\{\lambda_1, \lambda_2\}[A * I + I * B]^p = \max\{\lambda_1, \lambda_2\}(A\infty B)^p \end{aligned}$$

where, λ_1 and λ_2 are given in Eq (3.31) and Eq (3.32).

Similarly, from Eq (3.25), we obtain for

$$(A^p \infty B^p) - (A\infty B)^p \leq \max\{\alpha_1, \alpha_2\} I$$

where, α_1 and α_2 are given in Eq (3.34) and Eq (3.35). □

Special cases include from Eq (3.30):

(2.1) For $p = 2$, we have

$$A^2 \infty B^2 \leq \max\left\{\frac{(M_1 + m_1)^2}{4M_1 m_1}, \frac{(M_2 + m_2)^2}{4M_2 m_2}\right\}\{A\infty B\}^2. \tag{3.36}$$

(2.2) For $p = -1$, we have

$$A^{-1} \infty B^{-1} \leq \max\left\{\frac{(M_1 + m_1)^2}{4M_1 m_1}, \frac{(M_2 + m_2)^2}{4M_2 m_2}\right\}\{A\infty B\}^{-1}. \tag{3.37}$$

Similarly, special cases include from Eq (3.33):

(2.1) For $p = 2$, we have

$$(A^2 \infty B^2) - (A \infty B)^2 \leq \max \left\{ \frac{1}{4}(M_1 - m_1)^2, \frac{1}{4}(M_2 - m_2)^2 \right\} I. \tag{3.38}$$

(2.2) For $p = -1$, we have

$$(A^{-1} \infty B^{-1}) - (A \infty B)^{-1} \leq \max \left\{ \frac{\sqrt{M_1} - \sqrt{m_1}}{4M_1 m_1}, \frac{\sqrt{M_2} - \sqrt{m_2}}{4M_2 m_2} \right\} I. \tag{3.39}$$

Theorem 3.11. *Let A and B be positive definite Hermitian compatibly partitioned matrices. Let m and M be, respectively, the smallest and the largest eigenvalues of $A \nabla B$. Then*

(a) *For p a nonzero integer, we have*

$$A^p \infty B^p \leq \lambda (A \infty B)^p, \tag{3.40}$$

where λ is given by Eq (3.21).

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.40).

(b) *For p a nonzero integer, we have*

$$(A^p \infty B^p) - (A \infty B)^p \leq \alpha I \tag{3.41}$$

where, α is given by Eq (3.23).

While, for $0 < p < 1$, we have the reverse inequality in Eq (3.41).

Proof. In Eq (3.20) and Eq (3.22), take $k = 1$ and instead of U^*, use Z, the selection matrix which satisfy the following property:

$$A \infty B = Z'(A \nabla B) Z, \quad Z'Z = I$$

Then, with Z', $A \nabla B$, Z substituted for U, X, U^*, we have from Eq (3.20)

$$\begin{aligned} A^p \infty B^p &= Z'(A^p \nabla B^p) Z \\ &= Z'(A^p \Pi I + I \Pi B^p) Z \\ &\leq Z' \{A \nabla B\}^p Z \\ &\leq \lambda \{Z'(A \nabla B) Z\}^p = \lambda (A \infty B)^p \end{aligned}$$

where, λ is given by Eq (3.21). □

Similarly, from Eq (3.22), we obtain Eq (3.41)

Special cases include from Eq (3.40):

(2.1) For $p = 2$, we have

$$A^2 \infty B^2 \leq \frac{(M + m)^2}{4Mm} \{A \infty B\}^2 \tag{3.42}$$

(2.2) For $p = -1$, we have

$$A^{-1} \infty B^{-1} \leq \frac{(M + m)^2}{4Mm} \{A \infty B\}^{-1} \tag{3.43}$$

Similarly, special cases include from Eq (3.41):

(2.1) For $p = 2$, we have

$$(A^2 \infty B^2) - (A \infty B)^2 \leq \frac{1}{4}(M - m)^2 I \tag{3.44}$$

(2.2) For $p = -1$, we have

$$(A^{-1} \infty B^{-1}) - (A \infty B)^{-1} \leq \frac{\sqrt{M} - \sqrt{m}}{Mm} \{I\} \tag{3.45}$$

4. SPECIAL RESULTS ON HADAMARD AND KRONECKER SUMS

The results obtained in Section 3 are quite general. Now, we consider some inequalities in a special case which involves non-partitioned matrices A, B and I with the Hadamard product (sum) replacing the Khatri-Rao product (sum) and the Kronecker product (sum) replacing the Tracy-Singh product (sum). As these inequalities can be viewed as a corollary (some of) the proofs are straightforward and alternative to those for the existing inequalities.

Definition 4.1. Let A and B be square matrices of order $n \times n$.The *Hadamard sum* is defined as follows:

$$A \bullet B = A \circ I_n + I_n \circ B = A \circ I_n + B \circ I_n = (A + B) \circ I_n. \tag{4.1}$$

Corollary 4.1. *Let $A > 0$. Then*

$$A \bullet A^{-1} \geq 2I. \tag{4.2}$$

Corollary 4.2. *Let $A > 0$ be a matrix of order $n \times n$ with eigenvalues contained in the interval between m and M $(M \geq m)$. Then*

$$A \bullet A^{-1} \leq \frac{(m^2 + M^2)}{mM} \{I\}. \tag{4.3}$$

Corollary 4.3. *Let A and B be $n \times n$ positive definite Hermitian matrices and let m and M be, respectively, the smallest and the largest eigenvalues of $A \otimes B$. Then*

(a) *For p a nonzero integer, we have*

$$A^p \circ B^p \leq \lambda (A \circ B)^p \tag{4.4}$$

where, λ is given by Eq (3.21).

While, for $0 < p < 1$, we have the reverse inequality in Eq (4.4).

(b) *For p is a nonzero integer, we have*

$$(A^p \circ B^p) - (A \circ B)^p \leq \alpha I \tag{4.5}$$

where, α is given by Eq (3.23).

While, for $0 < p < 1$, we have the reverse inequality in Eq (4.5).

Special cases include from Eq (4.4):
(2.1) For $p = 2$, we have

$$A^2 \circ B^2 \leq \frac{(M + m)^2}{4Mm} \{A \circ B\}^2 \tag{4.6}$$

(2.2) For $p = -1$, we have

$$A^{-1} \circ B^{-1} \leq \frac{(M + m)^2}{4Mm} \{A \circ B\}^{-1}. \tag{4.7}$$

Similarly, special cases include from Eq (4.5):
(2.1) For $p = 2$, we have

$$(A^2 \circ B^2) - (A \circ B)^2 \leq \frac{1}{4}(M - m)^2 I \tag{4.8}$$

(2.2) For $p = -1$, we have

$$(A^{-1} \circ B^{-1}) - (A \circ B)^{-1} \leq \frac{\sqrt{M} - \sqrt{m}}{Mm} \{I\}, \tag{4.9}$$

where results in Eq (4.6), Eq (4.7), and Eq (4.8) are given in [11].

We note that the eigenvalues of $A \otimes B$ are the n^2 products of the eigenvalues of A by the eigenvalues of B.Thus if the eigenvalues of A and B are, respectively, ordered by:

$$\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n > 0, \quad \eta_1 \geq \eta_2 \geq \cdots \geq \eta_n > 0, \tag{4.10}$$

then in all the previous results in this section $M = \delta_1\eta_1$ and $m = \delta_n\eta_n$. Thus Eq (4.6) to Eq (4.9) become:

$$A^2 \circ B^2 \leq \frac{(\delta_1\eta_1 + \delta_n\eta_n)^2}{4\delta_1\eta_1\delta_n\eta_n} \{A \circ B\}^2 \tag{4.11}$$

$$A^{-1} \circ B^{-1} \leq \frac{(\delta_1\eta_1 + \delta_n\eta_n)^2}{4\delta_1\eta_1\delta_n\eta_n} \{A \circ B\}^{-1} \tag{4.12}$$

$$(A^2 \circ B^2) - (A \circ B)^2 \leq \frac{1}{4}(\delta_1\eta_1 - \delta_n\eta_n)^2 \{I\} \tag{4.13}$$

$$\left(A^{-1} \circ B^{-1}\right) - (A \circ B)^{-1} \leq \frac{\left(\sqrt{\delta_1\eta_1} - \sqrt{\delta_n\eta_n}\right)}{\delta_1\eta_1\delta_n\eta_n} \{I\} . \tag{4.14}$$

Corollary 4.4. *Let A and B be a $n \times n$ positive definite Hermitian matrices. Let m_1 and M_1 be, respectively, the smallest and the largest eigenvalues of $A \otimes I$ and m_2 and M_2, respectively, the smallest and the largest eigenvalues of $I \otimes B$. Then*

(a) *For p a nonzero integer, we have*

$$A^p \bullet B^p \leq \max\{\lambda_1, \lambda_2\} (A \bullet B)^p, \tag{4.15}$$

where λ_1 and λ_2 are given by Eq (3.31) and Eq (3.32).
While, for $0 < p < 1$, we have the reverse inequality in Eq (4.15).

(b) *For p a nonzero integer, we have*

$$(A^p \bullet B^p) - (A \bullet B)^p \leq \max\{\alpha_1, \alpha_2\} I, \tag{4.16}$$

where α_1 and $\alpha_2$are given by Eq (3.34) and Eq (3.35).
While, for $0 < p < 1$, we have the reverse inequality in Eq (4.16).

Note that, the eigenvalues of $A \otimes I$ equal the eigenvalues of A and the eigenvalues of $I \otimes B$ equal the eigenvalues of B.

Corollary 4.5. *Let A and B be $n \times n$ positive definite Hermitian matrices. Let m and M be, respectively, the smallest and the largest eigenvalues of $A \oplus B$. Then*

(a) *For p a nonzero integer, we have*

$$A^p \bullet B^p \leq \lambda(A \bullet B)^p, \tag{4.17}$$

where, λ is given by Eq (3.21).
While, for $0 < p < 1$, we have the reverse inequality in Eq (4.17).

(b) *For p a nonzero integer, we have*

$$(A^p \bullet B^p) - (A \bullet B)^p \leq \alpha I, \tag{4.18}$$

where, α is given by Eq (3.23).
While, for $0 < p < 1$, we have the reverse inequality in Eq (4.18).

Special cases include from Eq (4.17):
(2.1) For $p = 2$, we have

$$A^2 \bullet B^2 \leq \frac{(M+m)^2}{4Mm} \{A \bullet B\}^2 \tag{4.19}$$

(2.2) For $p = -1$, we have

$$A^{-1} \bullet B^{-1} \leq \frac{(M+m)^2}{4Mm} \{A \bullet B\}^{-1} \tag{4.20}$$

Similarly, special cases include from Eq (4.18):
(2.1) For $p = 2$, we have

$$(A^2 \bullet B^2) - (A \bullet B)^2 \leq \frac{1}{4}(M-m)^2 I \tag{4.21}$$

(2.2) For $p = -1$, we have

$$(A^{-1} \bullet B^{-1}) - (A \bullet B)^{-1} \leq \frac{\sqrt{M} - \sqrt{m}}{Mm} \{I\}. \tag{4.22}$$

We note that the eigenvalues of $A \oplus B$ are the n^2 sums of the eigenvalues of A by the eigenvalues of B. Thus if the eigenvalues of A and B are, respectively, ordered by:

$$\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n > 0, \quad \eta_1 \geq \eta_2 \geq \cdots \geq \eta_n > 0,$$

then in all previous results of this section $M = \delta_1 + \eta_1$ and $m = \delta_n + \eta_n$. Thus Eq(4.19) to Eq (4.22) become:

$$A^2 \bullet B^2 \leq \frac{(\delta_1 + \eta_1 + \delta_n + \eta_n)^2}{4(\delta_1 + \eta_1)(\delta_n + \eta_n)} \{A \bullet B\}^2, \tag{4.23}$$

$$A^{-1} \bullet B^{-1} \leq \frac{(\delta_1 + \eta_1 + \delta_n + \eta_n)^2}{4(\delta_1 + \eta_1)(\delta_n + \eta_n)} \{A \bullet B\}^{-1}, \tag{4.24}$$

$$(A^2 \bullet B^2) - (A \bullet B)^2 \leq \frac{1}{4}((\delta_1 + \eta_1) - (\delta_n + \eta_n))^2 I, \tag{4.25}$$

$$(A^{-1} \bullet B^{-1}) - (A \bullet B)^{-1} \leq \frac{\sqrt{\delta_1 + \eta_1} - \sqrt{\delta_n + \eta_n}}{(\delta_1 + \eta_1)(\delta_n + \eta_n)} I. \tag{4.26}$$

Corollary 4.6. *Let $A \geq 0$ and $B \geq 0$ be compatibly matrices. Then*

$$(i) \quad (A \oplus B)(A \oplus B)^* \geq AA^* \oplus BB^* \tag{4.27}$$

$$(ii) \quad (A \oplus B)^w \geq A^w \oplus B^w, \qquad \text{for any positive integer } w. \tag{4.28}$$

Corollary 4.7. *Let A and B be matrices of order $m \times m$ and $n \times n$ respectively. Then*

$$\text{(a)} \quad \operatorname{tr}(A \oplus B) = n \cdot \operatorname{tr}(A) + m \cdot \operatorname{tr}(B) \tag{4.29}$$

$$\text{(b)} \quad \|A \oplus B\|_p \leq \sqrt[p]{n}\, \|A\|_p + \sqrt[p]{m}\, \|B\|_p, \tag{4.30}$$

$$\text{where} \quad \|A\|_p = [\operatorname{tr} |A|^p]^{1/p},\ 1 \leq p < \infty.$$

$$\text{(c)} \quad e^{A \oplus B} = e^A \otimes e^B \tag{4.31}$$

Corollary 4.8. *Let A and B be non singular matrices of order $m \times m$ and $n \times n$, respectively. Then*

$$(i) \quad (A \oplus B)^{-1} = (A^{-1} \oplus B^{-1})^{-1}(A^{-1} \otimes B^{-1}) \tag{4.32}$$

$$(ii) \quad (A \oplus B)^{-1} = (A^{-1} \otimes I_n)(A^{-1} \oplus B^{-1})^{-1}(I_m \otimes B^{-1}) \tag{4.33}$$

$$(iii) \quad (A \oplus B)^{-1} = (I_m \otimes B^{-1})(A^{-1} \oplus B^{-1})^{-1}(A^{-1} \otimes I_n) \tag{4.34}$$

In [1], Ando proved the following inequality;

$$A \circ B \leq (A^p \circ I)^{\frac{1}{p}} (B^q \circ I)^{\frac{1}{q}}, \tag{4.35}$$

where A and B are positive definite matrices and $p, q \geq 1$with $1/p + 1/q = 1$.

If $\|\cdot\|$ is a unitarily invariant norm and $\|\cdot\|_\infty$ is the spectral norm, Horn and Johnson in [3] proved the following three conditions are equivalent:

$$\begin{array}{ll} \text{(i)} & \|A\|_\infty \leq \|A\| \\ \text{(ii)} & \|AB\| \leq \|A\| \cdot \|B\| \\ \text{(iii)} & \|A \circ B\| \leq \|A\| \cdot \|B\| \end{array} \tag{4.36}$$

for all matrices A and B.

In [2], Hiai and Zhan proved the following inequalities:

$$\frac{\|AB\|}{\|A\| \cdot \|B\|} \leq \frac{\|A+B\|}{\|A\| + \|B\|} \quad \text{and} \quad \frac{\|A \circ B\|}{\|A\| \cdot \|B\|} \leq \frac{\|A+B\|}{\|A\| + \|B\|} \tag{4.37}$$

for any invariant norm with $\|\operatorname{diag}(1, 0, \ldots, 0)\| \geq 1$ and A, B are nonzero positive definite matrices.

We have an application to generalize the inequalities in Eq (4.37) involving the Hadamard product (sum) and the Kronecker product (sum).

Theorem 4.9. *Let $\|\cdot\|$ be a unitarily invariant norm with $\|\operatorname{diag}(1, 0, \ldots, 0)\| \geq 1$ and A and B be nonzero positive definite matrices. Then*

$$\frac{\|A \circ B\|}{\|A\| \cdot \|B\|} \leq \frac{\|A \bullet B\|}{\|A\| + \|B\|}. \tag{4.38}$$

Proof. Let $\|\cdot\|_\infty$ be the spectral norm and applying Eq (4.35) to $A/\|A\|_\infty \leq I$, $B/\|B\|_\infty \leq I$ and using the Young inequality for scalars, we get

$$\begin{aligned} \frac{A}{\|A\|_\infty} \circ \left(\frac{B}{\|B\|_\infty}\right) &\leq \left[\left(\frac{A}{\|A\|_\infty}\right)^p \circ I\right]^{\frac{1}{p}} \left[\left(\frac{B}{\|B\|_\infty}\right)^q \circ I\right]^{\frac{1}{q}} \\ &\leq \frac{1}{p}\left(\frac{A}{\|A\|_\infty}\right)^p \circ I + \frac{1}{q}\left(\frac{B}{\|B\|_\infty}\right)^q \circ I \\ &\leq \frac{1}{p}\left(\frac{A}{\|A\|_\infty}\right) \circ I + \frac{1}{q}\left(\frac{B}{\|B\|_\infty}\right) \circ I \\ &= \left\{\frac{1}{p}\left(\frac{A}{\|A\|_\infty}\right) + \frac{1}{q}\left(\frac{B}{\|B\|_\infty}\right)\right\} \circ I \end{aligned}$$

We choose

$$\frac{1}{p} = \frac{\|A\|_\infty}{[\|A\|_\infty + \|B\|_\infty]} \quad \text{and} \quad \frac{1}{q} = \frac{\|B\|_\infty}{[\|A\|_\infty + \|B\|_\infty]}.$$

Since $\|A\|_\infty \le \|A\|$ and $\|B\|_\infty \le \|B\|$ thanks to$\|\mathrm{diag}(1,0,\dots,0)\| \ge 1$, we obtain

$$A \circ B \le \left\{ \frac{\|A\|_\infty \cdot \|B\|_\infty}{\|A\|_\infty + \|B\|_\infty} \right\} (A+B) \circ I \le \left\{ \frac{\|A\| \cdot \|B\|}{\|A\| + \|B\|} \right\} (A \bullet B) \tag{4.39}$$

Hence,

$$\|A \circ B\| \le \frac{\|A\| \cdot \|B\|}{\|A\| + \|B\|} \|A \bullet B\| \quad \text{or} \quad \frac{\|A \circ B\|}{\|A\| \cdot \|B\|} \le \frac{\|A \bullet B\|}{\|A\| + \|B\|}$$

□

Corollary 4.10. *Let $\|\cdot\|$ be a unitarily invariant norm with $\|\mathrm{diag}(1,0,\dots,0)\| \ge 1$ and A and B be nonzero positive definite matrices. Then*

$$\frac{\|A \otimes B\|}{\|A\| \cdot \|B\|} \le \frac{\|A \oplus B\|}{\|A\| + \|B\|}. \tag{4.40}$$

Proof. Applying Eq (2.7) and Eq (4.39), we have

$$K'(A \otimes B)K \le \frac{\|A\| \cdot \|B\|}{\|A\| + \|B\|} K'(A \oplus B)K$$

and

$$\|K'(A \otimes B)K\| \le \frac{\|A\| \cdot \|B\|}{\|A\| + \|B\|} \|K'(A \oplus B)K\|.$$

Provided that $\|\cdot\|$ is unitarily invariant norm, we get the result. □

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