Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 4, Article 113, 2004

SIMULTANEOUS APPROXIMATION BY LUPAŞ MODIFIED OPERATORS WITH WEIGHTED FUNCTION OF SZASZ OPERATORS

NAOKANT DEO

DEPARTMENT OF APPLIED MATHEMATICS
DELHI COLLEGE OF ENGINEERING
BAWANA ROAD, DELHI - 110042, INDIA.
dr_naokant_deo@yahoo.com

Received 20 August, 2004; accepted 01 September, 2004
Communicated by A. Lupaş

ABSTRACT. In the present paper, we consider a new modification of the Lupaş operators with the weight function of Szasz operators and study simultaneous approximation. Here we obtain a Voronovskaja type asymptotic formula and an estimate of error in simultaneous approximation for these Lupaş-Szasz operators.

Key words and phrases: Simultaneous approximation, Lupaş operators, Szasz operators.

2000 *Mathematics Subject Classification.* 41A28, 41A36.

1. INTRODUCTION

Lupaş proposed a family of linear positive operators mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $[0,\infty)$, namely,

$$(L_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

where $x \in [0,\infty)$.

Motivated by the integral modification of Bernstein polynomials by Derriennic [1], Sahai and Prasad [3] modified the operators L_n for functions integrable on $C[0,\infty)$ as

$$(M_n f)(x) = (n-1) \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} P_{n,k}(y) f(y) dy,$$

where

$$P_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}.$$

ISSN (electronic): 1443-5756

151-04

Integral modification of Szasz-Mirakyan operators were studied by Gupta [2]. Now we consider another modification of Lupaş operators with the weight function of Szasz operators, which are defined as

$$(B_n f)(x) = n \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} S_{n,k}(y) f(y) dy \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$$

and

$$S_{n,k}(y) = \frac{e^{-ny}(ny)^k}{k!}.$$

2. BASIC RESULTS

The following lemmas are useful for proving the main results.

Lemma 2.1. *Let* $m \in N^0$, $n \in \mathbb{N}$, *if we define*

$$T_{n,m}(x) = n \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy$$

then

(i) $T_{n,0}(x) = 1, T_{n,1}(x) = \frac{1+r(1+x)}{n}$, *and*

$$T_{n,2}(x) = \frac{rx(1+x) + 1 + [1 + r(1+x)]^2 + nx(2+x)}{n^2} \tag{2.1}$$

(ii) *For all* $x \geq 0$,

$$T_{n,m}(x) = O\left(\frac{1}{n^{\left[\frac{m+1}{2}\right]}}\right).$$

(iii)

$$nT_{n,m+1}(x) = x(1+x)T^{(1)}_{n,m+1}(x) + [m+1+r(1+x)]T_{n,m}(x) + mxT_{n,m-1}(x)$$

where $m \geq 2$.

Proof. The value of $T_{n,0}(x), T_{n,1}(x)$ easily follows from the definition, we give the proof of (iii) as follows.

$$\begin{aligned} x(x+1)T^{(1)}_{n,m}(x) &= n \sum_{k=0}^{\infty} x(1+x)P^{(1)}_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy \\ &\quad - mn \sum_{k=0}^{\infty} x(1+x)P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^{m-1} dy. \end{aligned}$$

Now using the identities

$$yS^{(1)}_{n,k}(y) = (k-ny)S_{n,k}(y),$$

and $x(1+x)P^{(1)}_{n,k}(x) = (k-nx)P_{n,k}(x)$, we get

$$\begin{aligned} &x(1+x)T^{(1)}_{n,m}(x) \\ &\quad = n \sum_{k=0}^{\infty} [k-(n+r)x]P_{n+r,k}(x) \int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy - mx(1+x)T_{n,m-1}(x). \end{aligned}$$

Therefore,

$$\begin{aligned}
&x(1+x)[T^{(1)}_{n,m}(x)+mT_{n,m-1}(x)]\\
&=n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty[(k+r-ny)+n(y-x)-r(1+x)]S_{n,k+r}(y)(y-x)^m dy\\
&=n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty yS^{(1)}_{n,k+r}(y)(y-x)^m dy+nT_{n,m+1}(x)-r(1+x)T_{n,m}(x)\\
&=n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty yS^{(1)}_{n,k+r}(y)(y-x)^{m+1} dy\\
&+nx\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty S^{(1)}_{n,k+r}(y)(y-x)^m dy+nT_{n,m+1}(x)-r(1+x)T_{n,m}(x)\\
&=-(m+1)T_{n,m}(x)-mxT_{n,m-1}(x)+nT_{n,m+1}(x)-r(1+x)T_{n,m}(x)
\end{aligned}$$

This leads to proof of (iii). □

Corollary 2.2. *Let α and δ be positive numbers, then for every $m\in\mathbb{N}$ and $x\in[0,\infty)$, there exists a positive constant $C_{m,x}$ depending on m and x such that*

$$n\sum_{k=0}^{\infty}P_{n,k}(x)\int\limits_{|t-x|\geq\delta}S_{n,k}(t)e^{\alpha t}dt\leq C_{m,x}n^{-m}.$$

Lemma 2.3. *If f is differentiable r times $(r=1,2,3,\dots)$ on $[0,\infty)$, then we have*

$$(B_n^{(r)}f)(x)=\frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty S_{n,k+r}(y)f^{(r)}(y)dy. \tag{2.2}$$

Proof. Applying Leibniz's theorem in (1.1) we have

$$\begin{aligned}
&(B_n^{(r)}f)(x)\\
&=n\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(n+k+r-i-1)!}{(n-1)!k!}(-1)^{r-i}x^{k-i}(1+x)^{-n-k-r+i}\int_0^\infty S_{n,k}(y)f(y)dy\\
&=n\sum_{k=0}^{\infty}\frac{(n+k+r-1)!}{(n-1)!k!}\cdot\frac{x^k}{(1+x)^{n+k+r}}\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}S_{n,k+i}(y)f(y)dy\\
&=n\frac{(n+r-1)!}{(n-1)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}S_{n,k+i}(y)f(y)dy.
\end{aligned}$$

Again using Leibniz's theorem,

$$\begin{aligned}
S^{(r)}_{n,k+r}(y)&=\sum_{i=0}^{r}\binom{r}{i}(-1)^i n^r\frac{e^{-ny}(ny)^{k+i}}{(k+i)!}\\
&=n^r\sum_{i=0}^{r}(-1)^i\binom{r}{i}S_{n,k+i}(y).
\end{aligned}$$

Hence

$$(B_n^{(r)}f)(x)=\frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty S^{(r)}_{n,k+r}(y)(-1)^r f(y)dy$$

and integrating by parts r times, we get the required result. □

3. MAIN RESULTS

Theorem 3.1. *Let f be integrable in $[0,\infty)$, admitting a derivative of order $(r+2)$ at a point $x \in [0,\infty)$. Also suppose $f^{(r)}(x) = o(e^{\alpha x})$ as $x \to \infty$, then*

$$\lim_{n\to\infty} n[(B_n^{(r)} f)(x) - f^{(r)}(x)] = [1 + r(1+x)]f^{(r+1)}(x) + x(2+x)f^{(r+2)}(x).$$

Proof. By Taylor's formula, we get

$$f^{(r)}(y) - f^{(r)}(x) = (y-x)f^{(r+1)}(x) + \frac{(y-x)^2}{2} f^{(r+2)}(x) + \frac{(y-x)^2}{2}\eta(y,x), \tag{3.1}$$

where

$$\eta(y,x) = \frac{f^{(r)}(y) - f^{(r)}(x) - (y-x)f^{(r+1)}(x) - \frac{(y-x)^2}{2} f^{(r+2)}(x)}{\frac{(y-x)^2}{2}} \quad \text{if} \quad x \neq y$$
$$= 0 \quad \text{if} \quad x = y.$$

Now, for arbitrary $\varepsilon > 0, \mathrm{A} > 0 \exists \quad a\delta > 0$ s. t.

$$|\eta(y,x)| \leq \varepsilon \quad \textit{for} \quad |y-x| < \delta, x \leq \mathrm{A}. \tag{3.2}$$

Using (2.2) in (3.1)

$$\begin{aligned}\frac{n^r (n-1)!}{(n+r-1)!}(B_n^{(r)} f)(x) - f^{(r)}(x) &= n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^\infty S_{n,k+r}(y) f^{(r)}(y)dy - f^{(r)}(x)\\ &= n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^\infty S_{n,k+r}(y)\left\{f^{(r)}(y) - f^{(r)}(x)\right\}dy\\ &= T_{n,1} f^{(r+1)}(x) + T_{n,2} f^{(r+2)}(x) + \mathrm{E}_{n,r}(x),\end{aligned}$$

where

$$\mathrm{E}_{n,r}(x) = \frac{n}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^\infty S_{n,k+r}(y)(y-x)^2\eta(y,x)dy.$$

In order to completely prove the theorem it is sufficient to show that

$$n\mathrm{E}_{n,r}(x) \to 0 \quad \textit{as} \quad n \to \infty.$$

Now

$$n\mathrm{E}_{n,r}(x) = R_{n,r,1}(x) + R_{n,r,2}(x),$$

where

$$R_{n,r,1}(x) = \frac{n^2}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|<\delta} S_{n,k+r}(y)(y-x)^2\eta(x,y)dy$$

and

$$R_{n,r,2}(x) = \frac{n^2}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|>\delta} S_{n,k+r}(y)(y-x)^2\eta(y,x)dy$$

By (3.2) and (2.1)

$$|R_{n,r,1}(x)| < \frac{n\varepsilon}{2}\left[n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|\leq\delta} S_{n,k+r}(y)(y-x)^2 dy\right] \leq \varepsilon x(2+x) \tag{3.3}$$

as $n \to \infty$.

Finally we estimate $R_{n,r,2}(x)$. Using Corollary 2.2 we have

$$R_{n,r,2}(x)=\frac{n^2}{2}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int\limits_{|y-x|>\delta}S_{n,k+r}(y)e^{\alpha y}dy=\frac{n}{2}M_{m,x}n^{-m}=0 \tag{3.4}$$

as $n\to\infty$. □

Theorem 3.2. *Let* $f\in C^{(r+1)}[0,a]$ *and let* $w(f^{(r+1)};\cdot)$ *be the modulus of continuity of* $f^{(r+1)}$*, then* $r=0,1,2,\dots$

$$\left\|(B_n^{(r)}f)(x)-f^{(r)}(x)\right\|\le\frac{[1+r(1+a)]}{n}\left\|f^{(r+1)}(x)\right\| +\frac{1}{n^2}\left(\sqrt{T_{n,2}(a)}+\frac{T_{n,2}(a)}{2}\right)w\left(f^{(r+1)};n^{-2}\right)$$

where $\|\cdot\|$ *is the sup norm* $[0,a]$.

Proof. We have by Taylor's expansion

$$\begin{aligned}
&f^{(r)}(y)-f^{(r)}(x)\\
&=(y-x)f^{(r+1)}(x)+\int_x^y[f^{(r+1)}(t)-f^{(r+1)}(x)]dt\frac{n^r(n-1)!}{(n+r-1)!}(B_n^{(r)}f)(x)-f^{(r)}(x)\\
&=n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}S_{n,k+r}(y)\left\{f^{(r)}(y)-f^{(r)}(x)\right\}dy\\
&=n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}S_{n,k+r}(y)\left((y-x)f^{(r+1)}(x)+\int_x^y[f^{(r+1)}(t)-f^{(r+1)}(x)]dt\right)dy.
\end{aligned}$$

Also

$$\left|f^{(r+1)}(t)-f^{(r+1)}(x)\right|\le\left(1+\frac{|t-x|}{\delta}\right)w(f^{(r+1)};\delta)$$

Hence

$$\left|\frac{n^r(n-1)!}{(n+r-1)!}(B_n^{(r)}f)(x)-f^{(r)}(x)\right| \le|T_{n,1}|\cdot\left|f^{(r+1)}(x)\right|+\left(\left|\sqrt{T_{n,2}}\right|+\frac{|T_{n,2}|}{2\delta}\right)\cdot w(f^{(r+1)};\delta).$$

By Schwarz's inequality. Choosing $\delta=\frac{1}{n^2}$ and using (i) and (2.1) we obtain the required result. □

REFERENCES

[1] M.M. DERRIENNIC, Sur l'approximation de fonctions integrables sur $[0,1]$ par des polynomes de Bernstein modifies, *J. Approx, Theory*, **31** (1981), 325–343.

[2] V. GUPTA, A note on modified Szasz operators, *Bull. Inst. Math. Academia Sinica*, **21**(3) (1993), 275–278.

[3] A. SAHAI AND G. PRASAD, On simultaneous approximation by modified Lupaş operators, *J. Approx., Theory*, **45** (1985), 122–128.