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SIMULTANEOUS APPROXIMATION BY LUPAŞ MODIFIED OPERATORS WITH WEIGHTED FUNCTION OF SZASZ OPERATORS

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Abstract

In the present paper, we consider a new modification of the Lupaş operators with the weight function of Szasz operators and study simultaneous approximation. Here we obtain a Voronovskaja type asymptotic formula and an estimate of error in simultaneous approximation for these Lupaş-Szasz operators.

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1. Introduction

Lupaş proposed a family of linear positive operators mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $[0,\infty)$, namely,

$$\left(L_n f\right)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

where $x \in [0,\infty)$.

Motivated by the integral modification of Bernstein polynomials by Derriennic [1], Sahai and Prasad [3] modified the operators L_n for functions integrable on $C[0,\infty)$ as

$$\left(M_n f\right)(x) = (n-1) \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} P_{n,k}(y) f(y)\,dy,$$

where

$$P_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}.$$

Integral modification of Szasz-Mirakyan operators were studied by Gupta [2]. Now we consider another modification of Lupaş operators with the weight function of Szasz operators, which are defined as

$$\left(B_n f\right)(x) = n \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} S_{n,k}(y) f(y)\,dy \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$$

and

$$S_{n,k}(y) = \frac{e^{-ny}(ny)^k}{k!}.$$

2. Basic Results

The following lemmas are useful for proving the main results.

Lemma 2.1. *Let* $m \in N^0$, $n \in \mathbb{N}$, *if we define*

$$T_{n,m}(x) = n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy$$

then

(i) $T_{n,0}(x) = 1, T_{n,1}(x) = \frac{1+r(1+x)}{n}$, *and*

$$T_{n,2}(x) = \frac{rx(1+x)+1+[1+r(1+x)]^2+nx(2+x)}{n^2} \tag{2.1}$$

(ii) For all $x \geq 0$,

$$T_{n,m}(x) = O\left(\frac{1}{n^{\left[\frac{m+1}{2}\right]}}\right).$$

(iii)

$$nT_{n,m+1}(x) = x(1+x)T^{(1)}_{n,m+1}(x)+[m+1+r(1+x)]T_{n,m}(x)+mxT_{n,m-1}(x)$$

where $m \geq 2$.

Proof. The value of $T_{n,0}(x), T_{n,1}(x)$ easily follows from the definition, we give the proof of (iii) as follows.

$$\begin{aligned}
&x(x+1)T^{(1)}_{n,m}(x)\\
&\quad = n\sum_{k=0}^{\infty} x(1+x)P^{(1)}_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy\\
&\qquad - mn\sum_{k=0}^{\infty} x(1+x)P_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)(y-x)^{m-1}dy.
\end{aligned}$$

Now using the identities

$$yS^{(1)}_{n,k}(y) = (k-ny)S_{n,k}(y),$$

and $x(1+x)P^{(1)}_{n,k}(x) = (k-nx)P_{n,k}(x)$, we get

$$\begin{aligned}
&x(1+x)T^{(1)}_{n,m}(x)\\
&\quad = n\sum_{k=0}^{\infty}[k-(n+r)x]P_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)(y-x)^m dy\\
&\qquad\qquad - mx(1+x)T_{n,m-1}(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&x(1+x)[T^{(1)}_{n,m}(x) + mT_{n,m-1}(x)]\\
&= n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}[(k+r-ny)+n(y-x)-r(1+x)]S_{n,k+r}(y)(y-x)^m dy
\end{aligned}$$

$$= n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}yS_{n,k+r}^{(1)}(y)(y-x)^m dy + nT_{n,m+1}(x) - r(1+x)T_{n,m}(x)$$

$$= n\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}yS_{n,k+r}^{(1)}(y)(y-x)^{m+1}dy$$

$$+ nx\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}S_{n,k+r}^{(1)}(y)(y-x)^m dy$$

$$+ nT_{n,m+1}(x) - r(1+x)T_{n,m}(x)$$

$$= -(m+1)T_{n,m}(x) - mxT_{n,m-1}(x) + nT_{n,m+1}(x) - r(1+x)T_{n,m}(x)$$

This leads to proof of (iii). □

Corollary 2.2. *Let α and δ be positive numbers, then for every $m \in \mathbb{N}$ and $x \in [0,\infty)$, there exists a positive constant $C_{m,x}$ depending on m and x such that*

$$n\sum_{k=0}^{\infty}P_{n,k}(x)\int_{|t-x|\geq\delta}S_{n,k}(t)e^{\alpha t}dt \leq C_{m,x}n^{-m}.$$

Lemma 2.3. *If f is differentiable r times $(r = 1,2,3,\dots)$ on $[0,\infty)$, then we have*

$$(B_n^{(r)}f)(x) = \frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}S_{n,k+r}(y)f^{(r)}(y)dy. \tag{2.2}$$

Proof. Applying Leibniz's theorem in (1.1) we have

$$
\begin{aligned}
(B_n^{(r)}f)(x) &= n\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(n+k+r-i-1)!}{(n-1)!k!}(-1)^{r-i}x^{k-i}(1+x)^{-n-k-r+i} \\
&\qquad\times\int_0^\infty S_{n,k}(y)f(y)dy \\
&= n\sum_{k=0}^{\infty}\frac{(n+k+r-1)!}{(n-1)!k!}\cdot\frac{x^k}{(1+x)^{n+k+r}}\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}S_{n,k+i}(y)f(y)dy \\
&= n\frac{(n+r-1)!}{(n-1)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}S_{n,k+i}(y)f(y)dy.
\end{aligned}
$$

Again using Leibniz's theorem,

$$
\begin{aligned}
S_{n,k+r}^{(r)}(y) &= \sum_{i=0}^{r}\binom{r}{i}(-1)^i n^r\frac{e^{-ny}(ny)^{k+i}}{(k+i)!} \\
&= n^r\sum_{i=0}^{r}(-1)^i\binom{r}{i}S_{n,k+i}(y).
\end{aligned}
$$

Hence

$$
(B_n^{(r)}f)(x) = \frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^\infty S_{n,k+r}^{(r)}(y)(-1)^r f(y)dy
$$

and integrating by parts r times, we get the required result. □

3. Main Results

Theorem 3.1. *Let f be integrable in $[0,\infty)$, admitting a derivative of order $(r+2)$ at a point $x \in [0,\infty)$. Also suppose $f^{(r)}(x) = o(e^{\alpha x})$ as $x \to \infty$, then*

$$\lim_{n\to\infty} n[(B_n^{(r)} f)(x) - f^{(r)}(x)] = [1 + r(1+x)]f^{(r+1)}(x) + x(2+x)f^{(r+2)}(x).$$

Proof. By Taylor's formula, we get

$$f^{(r)}(y) - f^{(r)}(x) = (y-x)f^{(r+1)}(x) + \frac{(y-x)^2}{2}f^{(r+2)}(x) + \frac{(y-x)^2}{2}\eta(y,x), \tag{3.1}$$

where

$$\eta(y,x) = \frac{f^{(r)}(y) - f^{(r)}(x) - (y-x)f^{(r+1)}(x) - \frac{(y-x)^2}{2}f^{(r+2)}(x)}{\frac{(y-x)^2}{2}} \quad \text{if} \quad x \neq y$$
$$= 0 \quad \text{if} \quad x = y.$$

Now, for arbitrary $\varepsilon > 0, \mathrm{A} > 0 \exists \quad a\delta > 0$ s. t.

$$|\eta(y,x)| \leq \varepsilon \quad \textit{for} \quad |y-x| < \delta, x \leq \mathrm{A}. \tag{3.2}$$

Using (2.2) in (3.1)

$$\frac{n^r(n-1)!}{(n+r-1)!}(B_n^{(r)}f)(x) - f^{(r)}(x)$$
$$= n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)f^{(r)}(y)dy - f^{(r)}(x)$$
$$= n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)\left\{f^{(r)}(y) - f^{(r)}(x)\right\}dy$$
$$= T_{n,1}f^{(r+1)}(x) + T_{n,2}f^{(r+2)}(x) + \mathrm{E}_{n,r}(x),$$

where

$$\mathrm{E}_{n,r}(x) = \frac{n}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^{\infty} S_{n,k+r}(y)(y-x)^2\eta(y,x)dy.$$

In order to completely prove the theorem it is sufficient to show that

$$n\mathrm{E}_{n,r}(x) \to 0 \quad \textit{as} \quad n \to \infty.$$

Now

$$n\mathrm{E}_{n,r}(x) = R_{n,r,1}(x) + R_{n,r,2}(x),$$

where

$$R_{n,r,1}(x) = \frac{n^2}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|<\delta} S_{n,k+r}(y)(y-x)^2\eta(x,y)dy$$

and

$$R_{n,r,2}(x) = \frac{n^2}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x) \int\limits_{|y-x|>\delta} S_{n,k+r}(y)(y-x)^2\eta(y,x)dy$$

By (3.2) and (2.1)

$$(3.3) \qquad |R_{n,r,1}(x)| < \frac{n\varepsilon}{2}\left[n\sum_{k=0}^{\infty} P_{n+r,k}(x) \int\limits_{|y-x|\le\delta} S_{n,k+r}(y)(y-x)^2 dy\right] \le \varepsilon x(2+x)$$

as $n \to \infty$.

Finally we estimate $R_{n,r,2}(x)$. Using Corollary 2.2 we have

$$(3.4) \qquad R_{n,r,2}(x) = \frac{n^2}{2}\sum_{k=0}^{\infty} P_{n+r,k}(x) \int\limits_{|y-x|>\delta} S_{n,k+r}(y)e^{\alpha y}dy = \frac{n}{2}M_{m,x}n^{-m} = 0$$

as $n \to \infty$. □

Theorem 3.2. *Let* $f \in C^{(r+1)}[0,a]$ *and let* $w(f^{(r+1)};\cdot)$ *be the modulus of continuity of* $f^{(r+1)}$*, then* $r = 0,1,2,\dots$

$$\left\|(B_n^{(r)}f)(x) - f^{(r)}(x)\right\| \le \frac{[1+r(1+a)]}{n}\left\|f^{(r+1)}(x)\right\| + \frac{1}{n^2}\left(\sqrt{T_{n,2}(a)} + \frac{T_{n,2}(a)}{2}\right) w\left(f^{(r+1)}; n^{-2}\right)$$

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where $\|\cdot\|$ *is the sup norm* $[0, a]$.

Proof. We have by Taylor's expansion

$$\begin{aligned} f^{(r)}(y) - f^{(r)}(x) &= (y-x)f^{(r+1)}(x) + \int_x^y [f^{(r+1)}(t) - f^{(r+1)}(x)]dt \\ &\qquad \times \frac{n^r(n-1)!}{(n+r-1)!}(B_n^{(r)}f)(x) - f^{(r)}(x) \\ &= n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^\infty S_{n,k+r}(y)\left\{f^{(r)}(y) - f^{(r)}(x)\right\}dy \\ &= n\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^\infty S_{n,k+r}(y)\Big((y-x)f^{(r+1)}(x) \\ &\qquad + \int_x^y [f^{(r+1)}(t) - f^{(r+1)}(x)]dt\Big)dy. \end{aligned}$$

Also

$$\left|f^{(r+1)}(t) - f^{(r+1)}(x)\right| \le \left(1 + \frac{|t-x|}{\delta}\right) w(f^{(r+1)};\delta)$$

Hence

$$\begin{aligned} &\left|\frac{n^r(n-1)!}{(n+r-1)!}(B_n^{(r)}f)(x) - f^{(r)}(x)\right| \\ &\qquad \le |T_{n,1}|\cdot\left|f^{(r+1)}(x)\right| + \left(\left|\sqrt{T_{n,2}}\right| + \frac{|T_{n,2}|}{2\delta}\right)\cdot w(f^{(r+1)};\delta). \end{aligned}$$

By Schwarz's inequality. Choosing $\delta = \frac{1}{n^2}$ and using (i) and (2.1) we obtain the required result. $\square$

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