

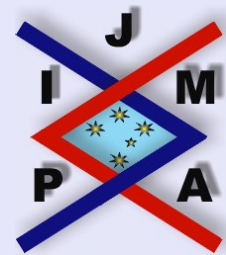
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AN ASYMPTOTIC EXPANSION

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Abstract

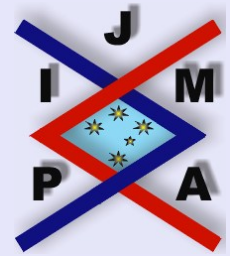
In this paper we study the asymptotic behaviour of the sequence $(r_n)_n$ of the powers of primes. Calculations also yield the evaluation $\sqrt{r_n} - p_n = o\left(\frac{n}{\log^s n}\right)$ for every positive integer s , p_n denoting the n -th prime.

2000 Mathematics Subject Classification: 11N05, 11N37

Key words: Powers of primes, Inequalities, Asymptotic behaviour

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1. Introduction

One denotes by:

- p_n the n -th prime
- r_n the n -th number (in increasing order) which can be written as a power p^m , $m \geq 2$, of a prime p .
- $\pi(x)$ the number of prime numbers not exceeding x .
- $\tilde{\pi}(x)$ the number of prime powers p^m , $m \geq 2$, not exceeding x .

The asymptotic equivalences

$$(1.1) \quad \pi(x) \sim \frac{x}{\log x}$$

and

$$(1.2) \quad p_n \sim n \log n$$

are well known.

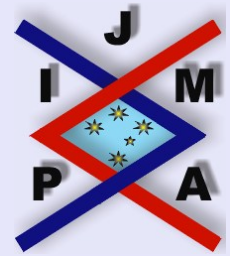
M. Cipolla [1] proves the relations

$$(1.3) \quad p_n = n(\log n + \log \log n - 1) + o(n)$$

and

$$(1.4) \quad p_n = n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} \right) + o \left(\frac{n}{\log n} \right)$$

that he generalizes to



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Theorem 1.1. *There exists a sequence $(P_m)_{m \geq 1}$ of polynomials with integer coefficients such that, for any integer m ,*

$$(1.5) \quad p_n = n \left[\log n + \log \log n - 1 + \sum_{j=1}^m \frac{(-1)^{j-1} P_j(\log \log n)}{\log^j n} + o\left(\frac{1}{\log^m n}\right) \right].$$

In the same paper, M. Cipolla gives recurrence formulae for P_m ; he finds that every P_m has degree m and leading coefficient $(m-1)!$.

As far as $(r_n)_n$ is concerned, L. Panaitopol [2] proves the asymptotic equivalence

$$(1.6) \quad r_n \sim n^2 \log^2 n.$$

We prove in this paper that $(r_n)_n$ has an asymptotic expansion comparable to that of Theorem 1.1.

We will need the next results of L. Panaitopol:

$$(1.7) \quad \tilde{\pi}(x) - \pi(\sqrt{x}) = O(\sqrt[3]{x}),$$

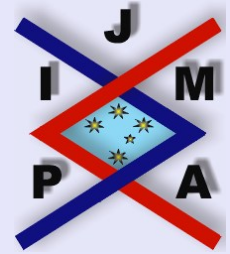
(from [2]), and

Proposition 1.2. *There exist a sequence of positive integers k_1, k_2, \dots and for every $n \geq 1$ a function α_n , $\lim_{x \rightarrow \infty} \alpha_n(x) = 0$, such that:*

$$(1.8) \quad \pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \dots - \frac{k_n(1+\alpha_n(x))}{\log^n x}}.$$

Moreover, k_1, k_2, \dots are given by the recurrence relation

$$(1.9) \quad k_n + 1! \cdot k_{n-1} + 2! \cdot k_{n-2} + \dots + (n-1)! \cdot k_1 = n \cdot n!, \quad n \geq 1.$$



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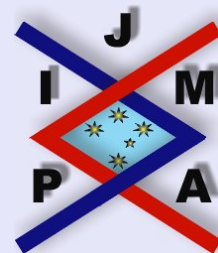
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(from [3]).

We will also establish a result similar to Proposition 1.2 for $\tilde{\pi}(x)$ and the evaluation

$$\sqrt{r_n} - p_n = o\left(\frac{n}{\log^s n}\right)$$

for every positive integer s .



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2. On the Asymptotic Behaviour of $\tilde{\pi}$

Proposition 2.1. For every positive integer n , there exists a function β_n , $\lim_{x \rightarrow \infty} \beta_n(x) = 0$, such that

$$(2.1) \quad \tilde{\pi}(x) = \frac{\sqrt{x}}{\log \sqrt{x} - 1 - \frac{k_1}{\log \sqrt{x}} - \dots - \frac{k_{n-1}}{\log^{n-1} \sqrt{x}} - \frac{k_n(1+\beta_n(x))}{\log^n \sqrt{x}}},$$

$(k_n)_n$ being the sequence of (1.9).

Proof. Let us set

$$(2.2) \quad \tilde{\pi}(x) = \frac{\sqrt{x}}{\log \sqrt{x} - 1 - \frac{k_1}{\log \sqrt{x}} - \dots - \frac{k_{n-1}}{\log^{n-1} \sqrt{x}} - \frac{k_n(1+\beta_n(x))}{\log^n \sqrt{x}}}.$$

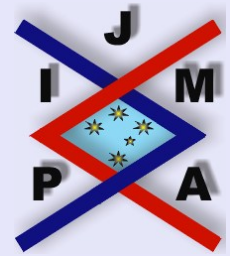
(1.8) and (1.7) give us:

$$(2.3) \quad \sqrt{x} \cdot \frac{k_n[\beta_n(x) - \alpha_n(\sqrt{x})]}{\log^{n+2} x} = O(\sqrt[3]{x}),$$

so

$$(2.4) \quad k_n[\beta_n(x) - \alpha_n(\sqrt{x})] = O\left(\frac{\log^{n+2} x}{\sqrt[6]{x}}\right),$$

leading to $\lim_{x \rightarrow \infty} \beta_n(x) = 0$. □



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3. Initial Estimates for r_n

Equation (2.1) gives:

$$(3.1) \quad \tilde{\pi}(x) \sim \frac{2\sqrt{x}}{\log x}.$$

If we put $x = r_n$, we get

$$(3.2) \quad n \sim \frac{2\sqrt{r_n}}{\log r_n},$$

so

$$(3.3) \quad \lim_{n \rightarrow \infty} (\log 2 + \log \sqrt{r_n} - \log n - \log \log r_n) = 0,$$

whence

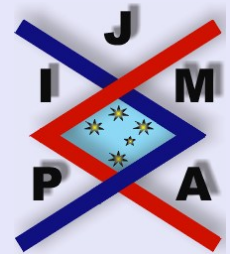
$$(3.4) \quad \lim_{n \rightarrow \infty} \frac{\log \sqrt{r_n}}{\log n} = 1,$$

leading to:

$$(3.5) \quad \lim_{n \rightarrow \infty} (\log \log r_n - \log 2 - \log \log n) = 0.$$

(3.3) and (3.5) give:

$$(3.6) \quad \log \sqrt{r_n} = \log n + \log \log n + o(1).$$



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(2.1) implies

$$(3.7) \quad \tilde{\pi}(x) = \frac{\sqrt{x}}{\log \sqrt{x} - 1 + o(1)}.$$

For $x = r_n$ we get (in view of (3.6)):

$$(3.8) \quad \frac{\sqrt{r_n}}{n} = \log n + \log \log n - 1 + o(1).$$

By taking logarithms of both sides we get:

$$(3.9) \quad \log \sqrt{r_n} - \log n = \log \log n + \log \left[1 + \frac{\log \log n - 1}{\log n} + o\left(\frac{1}{\log n}\right) \right].$$

For big enough n we have $\left| \frac{\log \log n - 1}{\log n} + o\left(\frac{1}{\log n}\right) \right| < 1$, which means that we can expand the logarithm. We derive:

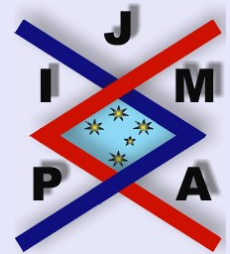
$$(3.10) \quad \log \sqrt{r_n} = \log n + \log \log n + \frac{\log \log n - 1}{\log n} + o\left(\frac{1}{\log n}\right).$$

(2.1) also gives:

$$(3.11) \quad \tilde{\pi}(x) = \frac{\sqrt{x}}{\log \sqrt{x} - 1 - \frac{1}{\log \sqrt{x}} + o\left(\frac{1}{\log x}\right)}.$$

For $x = r_n$ and in view of (3.4), we obtain:

$$(3.12) \quad \frac{\sqrt{r_n}}{n} = \log \sqrt{r_n} - 1 - \frac{1}{\log \sqrt{r_n}} + o\left(\frac{1}{\log n}\right).$$



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(3.10) and (3.12) allow us to write

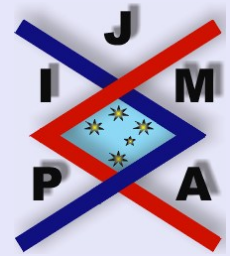
$$(3.13) \quad \frac{\sqrt{r_n}}{n} = \log n + \log \log n - 1 + \frac{\log \log n - 1}{\log n} - \frac{1}{\log n \left[1 + \frac{\log \log n}{\log n} + \frac{\log \log n - 1}{\log^2 n} + o\left(\frac{1}{\log^2 n}\right) \right]} + o\left(\frac{1}{\log n}\right).$$

For big enough n we have

$$\left| \frac{\log \log n}{\log n} + \frac{\log \log n - 1}{\log^2 n} + o\left(\frac{1}{\log^2 n}\right) \right| < 1;$$

we can therefore use the expansion of $\frac{1}{1+x}$ in (3.13) and we get

$$(3.14) \quad \sqrt{r_n} = n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} \right) + o\left(\frac{n}{\log n}\right).$$



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4. Main Result

Theorem 4.1. For every positive integer s we have

$$(4.1) \quad \frac{\sqrt{r_n} - p_n}{n} = o\left(\frac{1}{\log^s n}\right).$$

Proof. Induction with respect to s .

For $s = 1$ the statement is true because of (1.4) and (3.14).

Now let $s \geq 1$; suppose that

$$(4.2) \quad \frac{\sqrt{r_n} - p_n}{n} = o\left(\frac{1}{\log^s n}\right).$$

(4.2) and (1.5) lead to

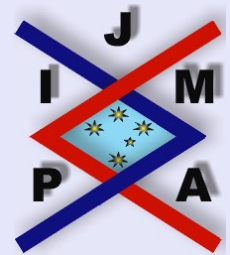
$$(4.3) \quad \sqrt{r_n} = n \left[\log n + \log \log n - 1 + \sum_{j=1}^s \frac{(-1)^{j-1} P_j(\log \log n)}{\log^j n} + o\left(\frac{1}{\log^s n}\right) \right].$$

By taking logarithms of both sides in (1.5) we derive

$$(4.4) \quad \log p_n = \log n + \log \log n + \log \left[1 + \frac{\log \log n - 1}{\log n} + \sum_{j=1}^s \frac{(-1)^{j-1} P_j(\log \log n)}{\log^{j+1} n} + o\left(\frac{1}{\log^{s+1} n}\right) \right].$$

(1.8) gives us

$$(4.5) \quad \pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \dots - \frac{k_{s+1}}{\log^{s+1} x} + o\left(\frac{1}{\log^{s+1} x}\right)}.$$



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For $x = p_n$, this relation becomes (in view of (1.2)):

$$(4.6) \quad \frac{p_n}{n} = \log p_n - 1 - \frac{k_1}{\log p_n} - \dots - \frac{k_{s+1}}{\log^{s+1} p_n} + o\left(\frac{1}{\log^{s+1} n}\right).$$

By taking logarithms of both sides in (4.3) we get

$$(4.7) \quad \log \sqrt{r_n} = \log n + \log \log n + \log \left[1 + \frac{\log \log n - 1}{\log n} + \sum_{j=1}^s \frac{(-1)^{j-1} P_j(\log \log n)}{\log^{j+1} n} + o\left(\frac{1}{\log^{s+1} n}\right) \right].$$

(2.1) gives

$$(4.8) \quad \tilde{\pi}(x) = \frac{x}{\log \sqrt{x} - 1 - \frac{k_1}{\log \sqrt{x}} - \frac{k_2}{\log^2 \sqrt{x}} - \dots - \frac{k_{s+1}}{\log^{s+1} \sqrt{x}} + o\left(\frac{1}{\log^{s+1} \sqrt{x}}\right)}.$$

For $x = r_n$, this relation becomes (in view of (3.4)):

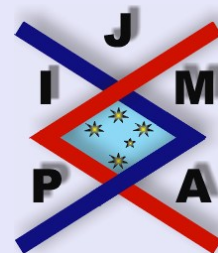
$$(4.9) \quad \frac{\sqrt{r_n}}{n} = \log \sqrt{r_n} - 1 - \frac{k_1}{\log \sqrt{r_n}} - \dots - \frac{k_{s+1}}{\log^{s+1} \sqrt{r_n}} + o\left(\frac{1}{\log^{s+1} n}\right).$$

If x and y are ≥ 1 , Lagrange's theorem gives us the inequality

$$(4.10) \quad |\log y - \log x| \leq |y - x|;$$

with (4.4) and (4.7), it leads to:

$$(4.11) \quad \log \sqrt{r_n} - \log p_n = o\left(\frac{1}{\log^{s+1} n}\right).$$



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This last relation gives for every $t \in \{1, 2, \dots, s+1\}$

$$(4.12) \quad \frac{1}{\log^t p_n} - \frac{1}{\log^t \sqrt{r_n}} = o\left(\frac{1}{\log^{s+t+2} n}\right) = o\left(\frac{1}{\log^{s+1} n}\right).$$

(4.6), (4.9), (4.11) and (4.12) give

$$(4.13) \quad \frac{\sqrt{r_n} - p_n}{n} = o\left(\frac{1}{\log^{s+1} n}\right)$$

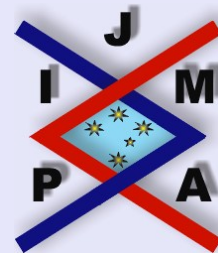
and the proof is complete. \square

Theorem 4.2. *There exists a unique sequence $(R_m)_{m \geq 1}$ of polynomials with integer coefficients such that, for every positive integer m ,*

$$(4.14) \quad r_n = n^2 \left[\log^2 n + 2(\log \log n - 1) \log n + (\log \log n)^2 - 3 + \sum_{j=1}^m \frac{(-1)^{j-1} R_j(\log \log n)}{(j+1)! \cdot \log^j n} \right] + o\left(\frac{n^2}{\log^m n}\right).$$

Proof. (4.9) allows us to write

$$(4.15) \quad r_n = n^2 \left[\log n + \log \log n - 1 + \sum_{j=1}^{m+1} \frac{(-1)^{j+1} P_j(\log \log n)}{j! \cdot \log^j n} + o\left(\frac{1}{\log^{m+1} n}\right) \right]^2.$$



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If we set

$$(4.16) \quad R_1 := 4(X - 1)P_1 - 2P_2$$

and

$$(4.17) \quad R_j := -2P_{j+1} + 2(j+1)(X-1)P_j - \sum_{i=1}^{j-1} (j+1) \binom{j}{i} P_i P_{j-i} \quad , j \geq 2$$

(4.15) gives for every $m \geq 1$:

$$r_n = n^2 \left[\log^2 n + 2(\log \log n - 1) \log n + (\log \log n)^2 - 3 + \sum_{j=1}^m \frac{(-1)^{j-1} R_j(\log \log n)}{(j+1)! \cdot \log^j n} \right] + o\left(\frac{n^2}{\log^m n}\right),$$

so the existence is proved.

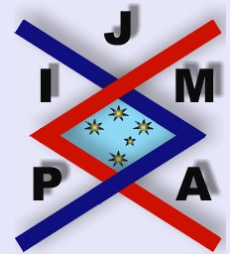
Suppose now the existence of two different sequences $(R_m)_{m \geq 1}$ and $(S_m)_{m \geq 1}$ satisfying the conditions of the theorem. For the least j such as $S_j \neq R_j$ we can write

$$\frac{R_j(\log \log n) - S_j(\log \log n)}{(j+1)! \cdot \log^j n} = o\left(\frac{1}{\log^j n}\right),$$

so $R_j(\log \log n) - S_j(\log \log n) = o(1)$, a contradiction. \square

Corollary 4.3. *We have*

$$r_n = n^2 \log^2 n + 2n^2(\log \log n - 1) \log n + n^2(\log \log n)^2 - 3n^2 + o(n^2).$$



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5. Computing the Coefficients of the Polynomial R_m

Proposition 5.1. For every $m \geq 1$, the degree of R_m is $m + 1$ and its leading coefficient is $2(m - 1)!$.

Proof. If we recall from the introduction that every P_n has degree n and leading coefficient $(n - 1)!$, the statement follows from (4.16) and (4.17). \square

(1.4) gives

$$P_1(X) = X - 2.$$

We can easily derive from M. Cipolla's paper [1] the relations

$$P'_k = k(k - 1)P_{k-1} + k \cdot P'_{k-1}, \quad k \geq 2$$

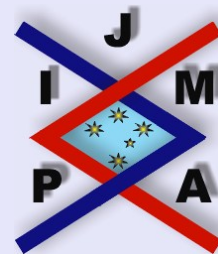
and

$$\begin{aligned} P_{k+1}(0) &= -k \left\{ \sum_{j=1}^{k-1} \binom{k-1}{j} P_j(0)[P_{k-j}(0) + P'_{k-j}(0)] + [P_k(0) + P'_k(0)] \right\} \\ &\quad - (k+1)P_k(0) - P'_{k+1}(0). \end{aligned}$$

Computations gave him

$$P_2(X) = X^2 - 6X + 11;$$

$$P_3(X) = 2X^3 - 21X^2 + 84X - 131;$$



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$$P_4(X) = 6X^4 - 92X^3 + 588X^2 - 1908X + 2666;$$

$$P_5(X) = 24X^5 - 490X^4 + 4380X^3 - 22020X^2 + 62860X - 81534;$$

$$P_6(X) = 120X^6 - 3084X^5 + 35790X^4 - 246480X^3 + 1075020X^2 \\ - 2823180X + 3478014;$$

$$P_7(X) = 720X^7 - 22428X^6 + 322224X^5 - 2838570X^4 + 16775640X^3 \\ - 66811920X^2 + 165838848X - 196993194.$$

In view of (4.16) and (4.17), we get in turn:

$$R_1(X) = 2X^2 - 14;$$

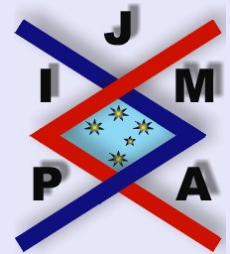
$$R_2(X) = 2X^3 - 6X^2 - 42X + 172;$$

$$R_3(X) = 4X^4 - 24X^3 - 144X^2 + 1544X - 3756;$$

$$R_4(X) = 12X^5 - 110X^4 - 600X^3 + 12300X^2 - 64060X + 122298;$$

$$R_5(X) = 48X^6 - 600X^5 - 2940X^4 + 102000X^3 - 842520X^2 \\ + 3319512X - 5484780;$$

$$R_6(X) = 240X^7 - 3836X^6 - 16380X^5 + 913080X^4 - 10543400X^3 \\ + 63989100X^2 - 215203884X + 323035480.$$



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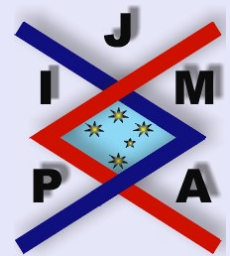
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