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BOUNDS FOR SOME PERTURBED ČEBYŠEV FUNCTIONALS

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ABSTRACT. Bounds for the perturbed Čebyšev functionals $C(f,g) - \mu C(e,g)$ and $C(f,g) - \mu C(e,g) - \nu C(f,e)$ when $\mu, \nu \in \mathbb{R}$ and e is the identity function on the interval $[a,b]$, are given. Applications for some Grüss' type inequalities are also provided.

Key words and phrases: Čebyšev functional, Grüss type inequality, Integral inequalities, Lebesgue $p-$norms.

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1. INTRODUCTION

For two Lebesgue integrable functions $f,g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1934, Grüss [5] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.4}$$

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provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ can be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a,b]$ while $\|f'\|_\infty = ess\sup_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [9]:

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is Lebesgue integrable and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of euclidean norms of the derivative was considered by A. Lupaş in [7] in which he proved that

$$|C(f,g)| \le \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [1] have proved the following results:

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \tag{1.7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} ess\sup_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|, \tag{1.8}$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ $(p > 1, \frac{1}{p} + \frac{1}{q} = 1; p = 1, q = \infty$ or $p = \infty, q = 1)$.

Notice that for $q = \infty, p = 1$ in (1.7) we obtain

$$\begin{aligned} |C(f,g)| &\le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\le \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \end{aligned} \tag{1.9}$$

and if g satisfies (1.3), then

$$\begin{aligned} |C(f,g)| &\le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\le \left\|g - \frac{n+N}{2}\right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\le \frac{1}{2}(N-n) \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt. \end{aligned} \tag{1.10}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [4]. However, the sharpness of the constant $\frac{1}{2}$, a generalisation for the abstract Lebesgue integral and the discrete version of it have been obtained in [2].

For other recent results on the Grüss inequality, see [6], [8] and [10] and the references therein.

The aim of the present paper is to establish Grüss type inequalities for some perturbed Čebyšev functionals. For this purpose, two integral representations of the functionals $C(f,g) - \mu C(e,g)$ and $C(f,g) - \mu C(e,g) - \nu C(f,e)$ when $\mu, \nu \in \mathbb{R}$ and $e(t) = t$, $t \in [a,b]$ are given.

2. REPRESENTATION RESULTS

The following representation result can be stated.

Lemma 2.1. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and g is Lebesgue integrable on $[a,b]$, then*

$$C(f,g) = \frac{1}{(b-a)^2} \int_a^b \int_a^b Q(t,s)\,[g(s) - \lambda]\, f'(t)\, ds dt \tag{2.1}$$

for any $\lambda \in \mathbb{R}$, where the kernel $Q : [a,b]^2 \to \mathbb{R}$ is given by

$$Q(t,s) := \begin{cases} t-b & \text{if } a \le s \le t \le b, \\ t-a & \text{if } a \le t < s \le b. \end{cases} \tag{2.2}$$

Proof. We observe that for $\lambda \in \mathbb{R}$ we have $C(f,\lambda) = 0$ and thus it suffices to prove (2.1) for $\lambda = 0$.

By Fubini's theorem, we have

$$\int_a^b \int_a^b Q(t,s)\, g(s)\, f'(t)\, ds dt = \int_a^b \left(\int_a^b Q(t,s)\, f'(t)\, dt \right) g(s)\, ds. \tag{2.3}$$

By the definition of $Q(t,s)$ and integrating by parts, we have successively,

$$\begin{aligned} \int_a^b Q(t,s)\, f'(t)\, dt &= \int_a^s Q(t,s)\, f'(t)\, dt + \int_s^b Q(t,s)\, f'(t)\, dt \\ &= \int_a^s (t-a)\, f'(t)\, dt + \int_s^b (t-b)\, f'(t)\, dt \\ &= (s-a)\, f(s) - \int_a^s f(t)\, dt + (b-s)\, f(s) - \int_s^b f(t)\, dt \\ &= (b-a)\, f(s) - \int_a^b f(t)\, dt, \end{aligned} \tag{2.4}$$

for any $s \in [a,b]$.

Now, integrating (2.4) multiplied with $g(s)$ over $s \in [a,b]$, we deduce

$$\begin{aligned} \int_a^b \left(\int_a^b Q(t,s)\, f'(t)\, dt \right) g(s)\, ds &= \int_a^b \left[(b-a)\, f(s) - \int_a^b f(t)\, dt \right] g(s)\, ds \\ &= (b-a) \int_a^b f(s) g(s) ds - \int_a^b f(s)\, ds \cdot \int_a^b g(s) ds \\ &= (b-a)^2\, C(f,g) \end{aligned}$$

and the identity is proved. □

Utilising the linearity property of $C(\cdot,\cdot)$ in each argument, we can state the following equality:

Theorem 2.2. *If* $e:[a,b]\to\mathbb{R}$, $e(t)=t$, *then under the assumptions of Lemma 2.1 we have:*

$$C(f,g)=\mu C(e,g)+\frac{1}{(b-a)^2}\int_a^b\int_a^b Q(t,s)[g(s)-\lambda][f'(t)-\mu]\,dtds \tag{2.5}$$

for any $\lambda,\mu\in\mathbb{R}$, *where*

$$C(e,g)=\frac{1}{b-a}\int_a^b tg(t)\,dt-\frac{a+b}{2}\int_a^b g(t)\,dt. \tag{2.6}$$

The second representation result is incorporated in

Lemma 2.3. *If* $f,g:[a,b]\to\mathbb{R}$ *are absolutely continuous on* $[a,b]$, *then*

$$C(f,g)=\frac{1}{(b-a)^2}\int_a^b\int_a^b K(t,s)f'(t)g'(s)\,dtds, \tag{2.7}$$

where the kernel $K:[a,b]\to\mathbb{R}$ *is defined by*

$$K(t,s):=\begin{cases}(b-t)(s-a) & \textit{if } a\le s\le t\le b,\\ (t-a)(b-s) & \textit{if } a\le t<s\le b.\end{cases} \tag{2.8}$$

Proof. By Fubini's theorem we have

$$\int_a^b\int_a^b K(t,s)f'(t)g'(s)\,dtds=\int_a^b\left(\int_a^b K(t,s)g'(s)\,ds\right)f'(t)\,dt. \tag{2.9}$$

By the definition of K and integrating by parts, we have successively:

$$\begin{aligned}\int_a^b K(t,s)g'(s)\,ds &=\int_a^t K(t,s)g'(s)\,ds+\int_t^b K(t,s)g'(s)\,ds\\ &=(b-t)\int_a^t (s-a)g'(s)\,ds+(t-a)\int_t^b(b-s)g'(s)\,ds\\ &=(b-t)\left[(t-a)g(t)-\int_a^t g(s)\,ds\right]\\ &\quad+(t-a)\left[-(b-t)g(t)+\int_t^b g(s)\,ds\right]\\ &=(t-a)\int_t^b g(s)\,ds-(b-t)\int_a^t g(s)\,ds,\end{aligned} \tag{2.10}$$

for any $t\in[a,b]$.

Multiplying (2.10) by $f'(t)$ and integrating over $t\in[a,b]$, we have:

$$\begin{aligned}\int_a^b\left(\int_a^b K(t,s)g'(s)\,ds\right)f'(t)\,dt &=\int_a^b\left[(t-a)\int_t^b g(s)\,ds-(b-t)\int_a^t g(s)\,ds\right]f'(t)\,dt\\ &=f(t)\left[(t-a)\int_t^b g(s)\,ds-(b-t)\int_a^t g(s)\,ds\right]\Bigg|_a^b\\ &\quad-\int_a^b f(t)\left[(t-a)\int_t^b g(s)\,ds-(b-t)\int_a^t g(s)\,ds\right]'dt\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&= \int_a^b f(t)\left[\int_t^b g(s)\,ds - (t-a)\,g(t) + \int_a^t g(s)\,ds - (b-t)\,g(t)\right] \\
&= -\int_a^b f(t)\left[\int_a^b g(s)\,ds - (b-a)\,g(t)\right] dt \\
&= (b-a)\int_a^b g(t)f(t)dt - \int_a^b f(t)\,dt \cdot \int_a^b g(t)dt \\
&= (b-a)^2\, C(f,g)\,.
\end{aligned}$$

By (2.11) and (2.9) we deduce the desired result. □

Theorem 2.4. *With the assumptions of Lemma 2.3, we have for any $\nu, \mu \in \mathbb{R}$ that:*

$$\begin{aligned}
C(f,g) &= \mu C(e,g) + \nu C(f,e) \\
&\quad + \frac{1}{(b-a)^2}\int_a^b\int_a^b K(t,s)\left[f'(t)-\mu\right]\left[g'(s)-\nu\right] dtds.
\end{aligned} \tag{2.12}$$

Proof. Follows by Lemma 2.3 on observing that $C(e,e) = 0$ and

$$C(f-\mu e, g-\nu e) = C(f,g) - \mu C(e,g) - \nu C(f,e)$$

for any $\mu, \nu \in \mathbb{R}$. □

3. Bounds in Terms of Lebesgue Norms of g and f'

Utilising the representation (2.5) we can state the following result:

Theorem 3.1. *Assume that $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$\begin{aligned}
&|C(f,g) - \mu C(e,g)| \\
&\leq \begin{cases}
\frac{1}{3}(b-a)\left\|f'-\mu\right\|_\infty \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty & \text{if } f', g \in L_\infty[a,b]\,; \\
\frac{2^{1/q}(b-a)^{\frac{p-q}{pq}}}{[(q+1)(q+2)]^{1/q}}\left\|f'-\mu\right\|_p \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_p & \text{if } f', g \in L_p[a,b]\,, \\
 & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
(b-a)^{-1}\left\|f'-\mu\right\|_1 \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 &
\end{cases}
\end{aligned} \tag{3.1}$$

for any $\mu \in \mathbb{R}$.

Proof. From (2.5), we have

$$\begin{aligned}
|C(f,g) - \mu C(e,g)| &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b |Q(t,s)|\,|g(s)-\lambda|\,|f'(t)-\mu|\,dtds \\
&\leq \|g-\lambda\|_\infty \left\|f'-\mu\right\|_\infty \frac{1}{(b-a)^2}\int_a^b\int_a^b |Q(t,s)|\,dtds.
\end{aligned} \tag{3.2}$$

However, by the definition of Q we have for $\alpha \geq 1$ that

$$\begin{aligned} I(\alpha) &:= \int_a^b \int_a^b |Q(t,s)|^\alpha \, dtds \\ &= \int_a^b \left(\int_a^t |t-b|^\alpha \, ds + \int_t^b |t-a|^\alpha \, ds \right) dt \\ &= \int_a^b \left[(t-a)(b-t)^\alpha + (b-t)(t-a)^\alpha \right] dt. \end{aligned}$$

Since

$$\int_a^b (t-a)(b-t)^\alpha \, dt = \frac{(b-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}$$

and

$$\int_a^b (b-t)(t-a)^\alpha \, dt = \frac{(b-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)},$$

hence

$$I(\alpha) = \frac{2(b-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}, \qquad \alpha \geq 1.$$

Then we have

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b |Q(t,s)| \, dtds = \frac{b-a}{3},$$

and taking the infimum over $\lambda \in \mathbb{R}$ in (3.2), we deduce the first part of (3.1).

Utilising the Hölder inequality for double integrals we also have

$$\begin{aligned} &\int_a^b \int_a^b |Q(t,s)| \, |g(s)-\lambda| \, |f'(t)-\mu| \, dtds \\ &\leq \left(\int_a^b \int_a^b |Q(t,s)|^q \, dtds \right)^{\frac{1}{q}} \left(\int_a^b \int_a^b |g(s)-\lambda|^p \, |f'(t)-\mu|^p \, dtds \right)^{\frac{1}{p}} \\ &= \frac{2^{1/q}(b-a)^{1+\frac{2}{q}}}{[(q+1)(q+2)]^{1/q}} \|g-\lambda\|_p \|f'-\mu\|_p , \end{aligned}$$

which provides, by the first inequality in (3.2), the second part of (3.1).

For the last part, we observe that $\sup_{(t,s)\in[a,b]^2} |Q(t,s)| = b-a$ and then

$$\int_a^b \int_a^b |Q(t,s)| \, |g(s)-\lambda| \, |f'(t)-\mu| \, dtds \leq (b-a) \|g-\lambda\|_1 \|f'-\mu\|_1 .$$

This completes the proof. □

Remark 1. The above inequality (3.1) is a source of various inequalities as will be shown in the following.

(1) For instance, if $-\infty < m \leq g(t) \leq M < \infty$ for a.e. $t \in [a,b]$, then $\left\|g - \frac{m+M}{2}\right\|_\infty \leq \frac{1}{2}(M-m)$ and $\left\|g - \frac{m+M}{2}\right\|_p \leq \frac{1}{2}(M-m)(b-a)^{1/p}$, $p \geq 1$. Then for any $\mu \in \mathbb{R}$ we have

$$(3.3) \qquad |C(f,g) - \mu C(e,g)| \leq \begin{cases} \frac{1}{6}(b-a)(M-m)\|f'-\mu\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}}(M-m)\|f'-\mu\|_p & \text{if } f' \in L_p[a,b], \\ & p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}(M-m)\|f'-\mu\|_1 , \end{cases}$$

which gives for $\mu = 0$ that

$$(3.4)\qquad |C(f,g)| \le \begin{cases} \frac{1}{6}(b-a)(M-m)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}}(M-m)\|f'\|_p & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(M-m)\|f'\|_1 . \end{cases}$$

(2) If $-\infty < \gamma \le f'(t) \le \Gamma < \infty$ for a.e. $t \in [a,b]$, then $\left\|f' - \frac{\gamma+\Gamma}{2}\right\|_\infty \le \frac{1}{2}|\Gamma-\gamma|$ and $\left\|f' - \frac{\gamma+\Gamma}{2}\right\|_p \le \frac{1}{2}|\Gamma-\gamma|(b-a)^{1/p}$, $p \ge 1$. Then we have from (3.1) that

$$(3.5)\qquad \left|C(f,g) - \frac{\gamma+\Gamma}{2}C(e,g)\right| \le \begin{cases} \frac{1}{6}(b-a)(\Gamma-\gamma)\inf_{\xi\in\mathbb{R}}\|g-\xi\|_\infty & \text{if } g \in L_\infty[a,b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}}(\Gamma-\gamma)\inf_{\xi\in\mathbb{R}}\|g-\xi\|_p & \text{if } g \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(\Gamma-\gamma)\inf_{\xi\in\mathbb{R}}\|g-\xi\|_1 . \end{cases}$$

Moreover, if we also assume that $-\infty < m \le g(t) \le M < \infty$ for a.e. $t \in [a,b]$, then by (3.5) we also deduce:

$$(3.6)\qquad \left|C(f,g) - \frac{\gamma+\Gamma}{2}C(e,g)\right| \le \begin{cases} \frac{1}{12}(b-a)(\Gamma-\gamma)(M-m) \\ \frac{2^{1-1/p}(b-a)}{[(q+1)(q+2)]^{1/q}}(\Gamma-\gamma)(M-m) & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}(\Gamma-\gamma)(M-m)(b-a) . \end{cases}$$

Observe that the first inequality in (3.6) is better than the others.

4. BOUNDS IN TERMS OF LEBESGUE NORMS OF f' AND g'

We have the following result:

Theorem 4.1. *Assume that $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$, then*

$$(4.1)\qquad |C(f,g) - \mu C(e,g) - \nu C(f,e)| \le \begin{cases} \frac{1}{12}(b-a)^2\|f'-\mu\|_\infty\|g'-\nu\|_\infty & \textit{if } f',g' \in L_\infty[a,b]; \\ \left[\frac{B(q+1,q+1)}{q+1}\right]^{\frac{1}{q}}(b-a)^{2/q}\|f'-\mu\|_p\|g'-\nu\|_p & \textit{if } f',g' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}\|f'-\mu\|_1\|g'-\nu\|_1 ; \end{cases}$$

for any $\mu, \nu \in \mathbb{R}$.

Proof. From (2.12), we have

$$|C(f,g)-\mu C(e,g)-\nu C(f,e)| \leq \frac{1}{(b-a)^2}\int_a^b\int_a^b |K(t,s)|\,|f'(t)-\mu|\,|g'(s)-\nu|\,dtds. \tag{4.2}$$

Define

$$\begin{aligned} J(\alpha) &:= \int_a^b\int_a^b |K(t,s)|^\alpha\,dtds \\ &= \int_a^b\left[\int_a^t (b-t)^\alpha (s-a)^\alpha\,ds + \int_t^b (t-a)^\alpha (b-s)^\alpha\,ds\right]dt \\ &= \frac{1}{\alpha+1}\left[\int_a^b (b-t)^\alpha (t-a)^{\alpha+1}\,dt + \int_a^b (t-a)^\alpha (b-t)^{\alpha+1}\,dt\right]. \end{aligned} \tag{4.3}$$

Since

$$\begin{aligned} \int_a^b (t-a)^p (b-t)^q\,dt &= (b-a)^{p+q+1}\int_0^1 s^p (1-s)^q\,ds \\ &= (b-a)^{p+q+1} B(p+1,q+1), \end{aligned}$$

hence, by (4.3),

$$J(\alpha) = \frac{2(b-a)^{2\alpha+2}}{\alpha+1} B(\alpha+1,\alpha+2), \qquad \alpha \geq 1.$$

As it is well known that

$$B(p,q+1) = \frac{q}{p+q} B(p,q),$$

then for $p=\alpha+1$, $q=\alpha+1$ we have $B(\alpha+1,\alpha+2) = \frac{1}{2}B(\alpha+1,\alpha+1)$.

Then we have

$$J(\alpha) = \frac{(b-a)^{2\alpha+2}}{\alpha+1} B(\alpha+1,\alpha+1), \qquad \alpha \geq 1.$$

Taking into account that

$$\begin{aligned} &\frac{1}{(b-a)^2}\int_a^b\int_a^b |K(t,s)|\,|f'(t)-\mu|\,|g'(s)-\nu|\,dtds \\ &\qquad \leq \|f'-\mu\|_\infty \|g'-\nu\|_\infty \frac{1}{(b-a)^2}\int_a^b\int_a^b |K(t,s)|\,dtds \\ &\qquad = \|f'-\mu\|_\infty \|g'-\nu\|_\infty (b-a)^2 B(2,3) \\ &\qquad = \frac{1}{12}(b-a)^2 \|f'-\mu\|_\infty \|g'-\nu\|_\infty, \end{aligned}$$

we deduce from (4.2) the first part of (4.1).

By the Hölder integral inequality for double integrals, we have

$$\begin{aligned}(4.4)\qquad &\int_a^b\int_a^b |K(t,s)|\,|f'(t)-\mu|\,|g'(s)-\nu|\,dtds\\ &\le \left(\int_a^b\int_a^b |K(t,s)|^q\,dtds\right)^{\frac{1}{q}}\|f'-\mu\|_p\,\|g'-\nu\|_p\\ &=\left[\frac{(b-a)^{2q+2}}{q+1}B(q+1,q+2)\right]^{\frac{1}{q}}\|f'-\mu\|_p\,\|g'-\nu\|_p\\ &=(b-a)^{2+2/q}\left[\frac{B(q+1,q+1)}{q+1}\right]^{\frac{1}{q}}\|f'-\mu\|_p\,\|g'-\nu\|_p\,.\end{aligned}$$

Utilising (4.2) and (4.4) we deduce the second part of (4.1).

By the definition of $K(t,s)$ we have, for $a\le s\le t\le b$, that

$$K(t,s)=(b-t)(s-a)\le(b-t)(t-a)\le\frac{1}{4}(b-a)^2$$

and for $a\le t<s\le b$, that

$$K(t,s)=(t-a)(b-s)\le(t-a)(b-t)\le\frac{1}{4}(b-a)^2\,,$$

therefore

$$\sup_{(t,s)\in[a,b]}|K(t,s)|=\frac{1}{4}(b-a)^2\,.$$

Due to the fact that

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b\int_a^b |K(t,s)|\,|f'(t)-\mu|\,|g'(s)-\nu|\,dtds\\ &\le \sup_{(t,s)\in[a,b]}|K(t,s)|\,\frac{1}{(b-a)^2}\int_a^b\int_a^b |f'(t)-\mu|\,|g'(s)-\nu|\,dtds\\ &=\frac{1}{4}\|f'-\mu\|_1\,\|g'-\nu\|_1\,,\end{aligned}$$

then from (4.2) we obtain the last part of (4.1). □

Remark 2. When $\mu=\nu=0$, we obtain from (4.1) the following Grüss type inequalities:

$$(4.5)\qquad |C(f,g)|\le\begin{cases}\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty & \text{if } f',g'\in L_\infty[a,b]\,;\\[2mm] \left[\frac{B(q+1,q+1)}{q+1}\right]^{\frac{1}{q}}(b-a)^{2/q}\|f'\|_p\|g'\|_p & \text{if } f',g'\in L_p[a,b]\,,\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{1}{4}\|f'\|_1\|g'\|_1\,. & \end{cases}$$

Notice that the first inequality in (4.5) is exactly the Čebyšev inequality for which $\frac{1}{12}$ is the best possible constant.

If we assume that there exists γ,Γ,ϕ,Φ such that $-\infty<\gamma\le f'(t)\le\Gamma<\infty$ and $-\infty<\phi\le g'(t)\le\Phi<\infty$ for a.e. $t\in[a,b]$, then we deduce from (4.1) the following inequality

$$(4.6)\qquad \left|C(f,g)-\frac{\gamma+\Gamma}{2}\cdot C(e,g)-\frac{\phi+\Phi}{2}\cdot C(f,e)\right|\le\frac{1}{48}(b-a)^2(\Gamma-\gamma)(\Phi-\phi)\,.$$

We also observe that the constant $\frac{1}{48}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

The sharpness of the constant follows by the fact that for $\Gamma = -\gamma$, $\Phi = -\phi$ we deduce from (4.6) the Čebyšev inequality which is sharp.

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