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ON THE HOMOGENEOUS FUNCTIONS WITH TWO PARAMETERS AND ITS MONOTONICITY

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ABSTRACT. Suppose $f(x,y)$ is a positive homogeneous function defined on $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, call $H_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}}$ homogeneous function with two parameters. If $f(x,y)$ is 2nd differentiable, then the monotonicity in parameters p and q of $H_f(a,b;p,q)$ depend on the signs of $I_1 = (\ln f)_{xy}$, for variable a and b depend on the sign of $I_{2a} = [(\ln f)_x \ln(y/x)]_y$ and $I_{2b} = [(\ln f)_y \ln(x/y)]_x$ respectively. As applications of these results, a serial of inequalities for arithmetic mean, geometric mean, exponential mean, logarithmic mean, power-Exponential mean and exponential-geometric mean are deduced.

Key words and phrases: Homogeneous function with two parameters, f-mean with two-parameter, Monotonicity, Estimate for lower and upper bounds.

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1. INTRODUCTION

The so-called two-parameter mean or extended mean values between two unequal positive numbers a and b were defined first by K.B. Stolarsky in [10] as

$$E(a,b;p,q) = \begin{cases} \left(\frac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ \left(\frac{a^p-b^p}{p(\ln a - \ln b)}\right)^{\frac{1}{p}} & p \neq 0, q = 0 \\ \left(\frac{a^q-b^q}{q(\ln a - \ln b)}\right)^{\frac{1}{q}} & p = 0, q \neq 0 \\ \exp\left(\frac{a^p \ln a - b^p \ln b}{a^p - b^p} - \frac{1}{p}\right) & p = q \neq 0 \\ \sqrt{ab} & p = q = 0 \end{cases}. \tag{1.1}$$

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The monotonicity of $E(a,b;p,q)$ has been researched by E. B. Leach and M. C. Sholander in [4], and others also in [9, 8, 7, 6, 5, 11, 14, 15, 17] using different ideas and simpler methods.

As the generalized power-mean, C. Gini obtained a similar two-parameter type mean in [1]. That is:

$$G(a,b;p,q)=\begin{cases}\left(\frac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & p\neq q\\ \exp\left(\frac{a^p\ln a+b^p\ln b}{a^p+b^p}\right) & p=q\neq 0\\ \sqrt{ab} & p=q=0\end{cases}. \tag{1.2}$$

Recently, the sufficient and necessary conditions comparing the two-parameter mean with the Gini mean were put forward by using the so-called concept of "strong inequalities" ([3]).

From the above two-parameter type means, we find that their forms are both $\left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}$, where $f(x,y)$ is a homogeneous function of x and y.

The main aim of this paper is to establish the concept of "two-parameter homogeneous functions", and study the monotonicity of functions in the form of $\left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}$. As applications of the main results, we will deduce three inequality chains which contain the arithmetic, geometric, exponential, logarithmic, power-exponential and exponential-geometric means, prove an upper bound for the Stolarsky mean in [12], and present two estimated expressions for the exponential mean.

2. Basic Concepts and Main Results

Definition 2.1. Assume that $f:\mathbb{U}(\subseteq\mathbb{R}_+\times\mathbb{R}_+)\to\mathbb{R}_+$ is a homogeneous function of variable x and y, and is continuous and exists first order partial derivative, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ with $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$. If $(1,1)\notin\mathbb{U}$, then we define

$$\mathcal{H}_f(a,b;p,q)=\left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}}\qquad (p\neq q, pq\neq 0), \tag{2.1}$$

$$\mathcal{H}_f(a,b;p,p)=\lim_{q\to p}\mathcal{H}_f(a,b;p,q)=G_{f,p}(a,b)\qquad (p=q\neq 0), \tag{2.2}$$

where

$$G_{f,p}(a,b)=G_f^{\frac{1}{p}}(a^p,b^p),\, G_f(x,y)=\exp\left[\frac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}\right], \tag{2.3}$$

$f_x(x,y)$ and $f_y(x,y)$ denote partial derivative to 1st and 2nd variable of $f(x,y)$ respectively.

If $(1,1)\in\mathbb{U}$, then define further

$$\mathcal{H}_f(a,b;p,0)=\left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}}\qquad (p\neq 0, q=0), \tag{2.4}$$

$$\mathcal{H}_f(a,b;0,q)=\left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}}\qquad (p=0, q\neq 0), \tag{2.5}$$

$$\mathcal{H}_f(a,b;0,0)=\lim_{p\to 0}\mathcal{H}_f(a,b;p,0)=a^{\frac{f_x(1,1)}{f(1,1)}}b^{\frac{f_y(1,1)}{f(1,1)}}\qquad (p=q=0). \tag{2.6}$$

From Lemma 3.1, $\mathcal{H}_f(a,b;p,q)$ is still a homogeneous function of positive numbers a and b. We call it a homogeneous function for positive numbers a and b with two parameters p and q, and call it a two-parameter homogeneous function for short. To avoid confusion, we also denote it by $\mathcal{H}_f(p,q)$ or $\mathcal{H}_f(a,b)$ or $\mathcal{H}_f$.

If $f(x,y)$ is a positive 1-order homogeneous mean function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, then call $\mathcal{H}_f(a,b;p,q)$ the two-parameter f-mean of positive numbers a and b.

Remark 2.1. If $f(x,y)$ is a positive 1-order homogeneous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, and is continuous and exists 1st order partial derivative, and satisfies $f(x,y) = f(y,x)$, then

$$G_{f,0}(a,b) = \mathcal{H}_f(a,b;0,0) = \sqrt{ab}.$$

In fact, by (2.3), we have

$$G_{f,0}(a,b) = \exp\left[\frac{f_x(1,1)\ln a + f_y(1,1)\ln b}{f(1,1)}\right] = \mathcal{H}_f(a,b;0,0).$$

Since $f(x,y)$ is a positive 1-order homogeneous function, from (3.1) of Lemma 3.2, we obtain

$$\frac{1\cdot f_x(1,1)}{f(1,1)} + \frac{1\cdot f_y(1,1)}{f(1,1)} = 1. \tag{2.7}$$

If $f(x,y) = f(y,x)$, then $f_x(x,y) = f_y(y,x)$, so we have

$$f_x(1,1) = f_y(1,1). \tag{2.8}$$

By (2.7) and (2.8), we get

$$\frac{f_x(1,1)}{f(1,1)} = \frac{f_y(1,1)}{f(1,1)} = \frac{1}{2},$$

thereby $G_{f,0} = \sqrt{ab}$.

Thus it can be seen that despite the form of $f(x,y)$ we always have $\mathcal{H}_f(a,b;0,0) = G_{f,0}(a,b) = \sqrt{ab}$, so long as $f(x,y)$ is a positive 1-order homogeneous symmetric function defined on $\mathbb{R}_+ \times \mathbb{R}_+$.

Example 2.1. In Definition 2.1, let $f(x,y) = L(x,y) = \frac{x-y}{\ln x - \ln y}$ $(x,y > 0, x \neq y)$, we get (1.1), i.e.

$$\mathcal{H}_L(a,b;p,q) = \begin{cases} \left(\frac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ L^{\frac{1}{p}}(a^p,b^p) & p \neq 0, q = 0 \\ L^{\frac{1}{q}}(a^q,b^q) & p = 0, q \neq 0 \\ G_{L,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{2.9}$$

where

$$G_{L,p}(a,b) = E_p(a,b) = E^{\frac{1}{p}}(a^p,b^p) = E_p,$$

$$E(a,b) = e^{-1}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, \qquad G(a,b) = \sqrt{ab}.$$

Remark 2.2. That

$$E(a,b) = e^{-1}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}} \qquad (a,b > 0 \text{ with } a \neq b)$$

is called the exponential mean of unequal positive numbers a and b, and is also called the identical mean and denoted by $I(a,b)$. To avoid confusion, we adopt our terms and notations in what follows.

Example 2.2. In Definition 2.1, let $f(x,y) = A(x,y) = \frac{x+y}{2}$ $(x,y>0, x\neq y)$, we get (1.2), i.e.

$$\mathcal{H}_A(a,b;p,q) = \begin{cases} \left(\frac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & p\neq q \\ G_{A,p}(a,b) & p=q\neq 0 \\ G(a,b) & p=q=0 \end{cases}, \tag{2.10}$$

where $G_{A,p}(a,b) = Z_p(a,b) = Z^{\frac{1}{p}}(a^p,b^p) = Z_p$. $Z(a,b) = a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$ is called the power-exponential mean between positive numbers a and b.

Example 2.3. In Definition 2.1, let $f(x,y) = E(x,y) = e^{-1}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$ $(x,y>0,x\neq y)$, then

$$\mathcal{H}_E(a,b;p,q) = \begin{cases} \left(\frac{E(a^p,b^p)}{E(a^q,b^q)}\right)^{\frac{1}{p-q}} & p\neq q \\ G_{E,p}(a,b) & p=q\neq 0 \\ G(a,b) & p=q=0 \end{cases}, \tag{2.11}$$

where $G_{E,p}(a,b) = Y_p(a,b) = Y^{\frac{1}{p}}(a^p,b^p) = Y_p$. $Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$ is called the exponential-geometric mean between positive numbers a and b, where $E = E(a,b)$, $L = L(a,b)$, $G = G(a,b)$.

Example 2.4. In Definition 2.1, let $f(x,y) = D(x,y) = |x-y|$ $(x,y>0,x\neq y)$, then

$$\mathcal{H}_D(a,b;p,q) = \begin{cases} \left|\frac{a^p-b^p}{a^q-b^q}\right|^{\frac{1}{p-q}} & p\neq q,\ pq\neq 0 \\ G_{D,p}(a,b) & p=q\neq 0 \end{cases}, \tag{2.12}$$

where $G_{D,p}(a,b) = G_{D,p} = e^{\frac{1}{p}}E^{\frac{1}{p}}(a^p,b^p) = e^{\frac{1}{p}}E_p$.

In order to avoid confusion, we rename $\mathcal{H}_L(a,b;p,q)$ (or $E(a,b;p,q)$) and $\mathcal{H}_A(a,b;p,q)$ (or $G(a,b;p,q)$) as the two-parameter logarithmic mean and two-parameter arithmetic mean respectively. In the same way, we call $\mathcal{H}_E(a,b;p,q)$ in Example 2.3 the two-parameter exponential mean.

In Example 2.4, since $D(x,y) = |x-y|$ is not a certain mean between positive numbers x and y, but one absolute value function of difference of two positive numbers, we call $\mathcal{H}_D(a,b;p,q)$ a two-parameter homogeneous function of difference.

It is obvious that the conception of two-parameter homogeneous functions has greatly developed the extension of the concept of two-parameter means.

For monotonicity of two-parameter homogeneous functions $\mathcal{H}_f(a,b;p,q)$, we have the following main results.

Theorem 2.3. *Let $f(x,y)$ be a positive $n-$order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be second order differentiable. If $I_1 = (\ln f)_{xy} > (<)0$, then $\mathcal{H}_f(p,q)$ is strictly increasing (decreasing) in both p and q on $(-\infty,0)\cup(0,+\infty)$.*

Corollary 2.4.

(1) *$\mathcal{H}_L(p,q), \mathcal{H}_A(p,q), \mathcal{H}_E(p,q)$ are strictly increasing both p and q on $(-\infty,+\infty)$,*
(2) *$\mathcal{H}_D(p,q)$ is strictly decreasing both p and q on $(-\infty,0)\cup(0,+\infty)$.*

Theorem 2.5. *Let $f(x,y)$ be a positive 1-order homogeneous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be second order differentiable.*

(1) *If* $I_{2a} = \left[(\ln f)_x \ln(y/x)\right]_y > (<)0$, *then* $\mathcal{H}_f(a,b)$ *is strictly increasing (decreasing) in* a.
(2) *If* $I_{2b} = \left[(\ln f)_y \ln(x/y)\right]_x > (<)0$, *then* $\mathcal{H}_f(a,b)$ *is strictly increasing (decreasing) in* b.

Corollary 2.6. $\mathcal{H}_L(a,b), \mathcal{H}_D(a,b)$ *is strictly increasing in both* a *and* b.

3. LEMMAS AND PROOFS OF THE MAIN RESULTS

For proving the main results in this article, we need some properties of homogeneous functions in [16]. For convenience, we quote them as follows.

Lemma 3.1. *Let* $f(x,y)$, $g(x,y)$ *be* n, m*-order homogenous functions over* Ω *respectively, then* $f \cdot g,\ f/g\ (g \neq 0)$ *are* $n+m, n-m$*-order homogenous functions over* Ω *respectively.*

If for a certain p *with* $(x^p, y^p) \in \Omega$, *and* $f^p(x,y)$ *exists, then* $f(x^p, y^p), f^p(x,y)$ *are both* np*-order homogeneous functions over* Ω.

Lemma 3.2. *Let* $f(x,y)$ *be a* n*-order homogeneous function over* Ω, *and* f_x, f_y *both exist, then* f_x, f_y *are both* $(n-1)$*-order homogeneous functions over* Ω, *furthermore we have*

$$xf_x + yf_y = nf. \tag{3.1}$$

In particular, when $n=1$ *and* $f(x,y)$ *is second order differentiable over* Ω, *then*

$$xf_x + yf_y = f, \tag{3.2}$$

$$xf_{xx} + yf_{xy} = 0, \tag{3.3}$$

$$xf_{xy} + yf_{yy} = 0. \tag{3.4}$$

Lemma 3.3. *Let* $f(x,y)$ *be a positive* $n-$*order homogenous function defined on* $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, *and be second order differentiable. Set*

$$T(t) = \ln f(a^t, b^t), \text{ where } x = a^t, y = b^t, a, b > 0,$$

then

$$T''(t) = -xyI_1(\ln b - \ln a)^2, \text{ where } I_1 = \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f)_{xy}.$$

Proof. Since $f(x,y)$ is a positive n-order homogeneous function, from (3.1), we can obtain $x(\ln f)_x + y(\ln f)_y = n$ or $x(\ln f)_x = n - y(\ln f)_y$, $y(\ln f)_y = n - x(\ln f)_x$, so

$$T'(t) = \frac{a^t f_x(a^t, b^t)\ln a + b^t f_y(a^t, b^t)\ln b}{f(a^t, b^t)} \tag{3.5}$$

$$= \frac{xf_x(x,y)\ln a + yf_y(x,y)\ln b}{f(x,y)} \tag{3.6}$$

$$= x(\ln f)_x \ln a + y(\ln f)_y \ln b. \tag{3.7}$$

Hence

$$\begin{aligned}
T''(t) &= \frac{\partial T'(t)}{\partial x}\frac{dx}{dt} + \frac{\partial T'(t)}{\partial y}\frac{dy}{dt} \\
&= \left[y(\ln f)_y(\ln b - \ln a) + n\ln a\right]_x a^t \ln a \\
&\qquad + \left[x(\ln f)_x(\ln a - \ln b) + n\ln b\right]_y b^t \ln b \\
&= y(\ln f)_{yx}(\ln b - \ln a)x\ln a + x(\ln f)_{xy}(\ln a - \ln b)y\ln b \\
&= -xy(\ln f)_{xy}(\ln b - \ln a)^2 \\
&= -xyI_1(\ln b - \ln a)^2.
\end{aligned}$$

$\square$

Lemma 3.4. *Let* $f(x,y)$ *be a positive* 1*-order homogeneous function defined on* $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$*, and be second order differentiable. Set*

$$S(t) = \frac{txf_x(x,y)}{f(x,y)}, \text{ where } x = a^t, y = b^t \ a,b > 0,$$

then

$$S'(t) = xyI_{2a}, \text{ where } I_{2a} = \left[(\ln f)_x \ln(y/x)\right]_y .$$

Proof.

$$\begin{aligned} S'(t) &= \frac{xf_x(x,y)}{f(x,y)} + t\frac{d}{dt}\left[\frac{xf_x(x,y)}{f(x,y)}\right] \\ &= x(\ln f)_x + t\left[\frac{\partial(x(\ln f)_x)}{\partial x}\frac{dx}{dt} + \frac{\partial(x(\ln f)_x)}{\partial y}\frac{dy}{dt}\right] \\ &= x(\ln f)_x + t\left[\frac{\partial(x(\ln f)_x)}{\partial x}a^t\ln a + \frac{\partial(x(\ln f)_x)}{\partial y}b^t\ln b\right] \\ &= x(\ln f)_x + t\left[x(x(\ln f)_x)_x \ln a + y(x(\ln f)_x)_y \ln b\right]. \end{aligned}$$

By Lemma 3.1, that $x(\ln f)_x = \frac{xf_x(x,y)}{f(x,y)}$ is a 0-order homogeneous function, from (3.1) of Lemma 3.2, we obtain $x\left[x(\ln f)_x\right]_x + y\left[x(\ln f)_x\right]_y = 0$ or $x\left[x(\ln f)_x\right]_x = -y\left[x(\ln f)_x\right]_y$, hence

$$\begin{aligned} S'(t) &= x(\ln f)_x + ty\left[x(\ln f)_x\right]_y(\ln b - \ln a) \\ &= x(\ln f)_x + txy(\ln f)_{xy}(\ln b - \ln a) \\ &= x(\ln f)_x + xy(\ln f)_{xy}(\ln b^t - \ln a^t) \\ &= x(\ln f)_x + xy(\ln f)_{xy}(\ln y - \ln x) \\ &= xy\left[y^{-1}(\ln f)_x + (\ln f)_{xy}\ln(y/x)\right] \\ &= xy\left[(\ln f)_x \ln(y/x)\right]_y = xyI_{2a}. \end{aligned}$$

$\square$

Based on the above lemmas, then next we will go on proving the main results in this paper.

Proof of Theorem 2.3. Since $\mathcal{H}_f(p,q)$ is symmetric with respect to p and q, we only need to prove the monotonicity for p of $\ln\mathcal{H}_f$.
1) When $p \neq q$,

$$\ln\mathcal{H}_f = \frac{1}{p-q}\ln\frac{f(a^p,b^p)}{f(a^q,b^q)} = \frac{T(p)-T(q)}{p-q},$$

$$\frac{\partial\ln\mathcal{H}_f}{\partial p} = \frac{(p-q)T'(p)-T(p)+T(q)}{(p-q)^2}.$$

Set $g(p) = (p-q)T'(p) - T(p) + T(q)$, then $g(q) = 0$, $g'(p) = (p-q)T''(p)$, and then exist $\xi = q + \theta(p-q)$ with $\theta \in (0,1)$ by Mean-value Theorem, such that

$$\frac{\partial\ln\mathcal{H}_f}{\partial p} = \frac{g(p)-g(q)}{(p-q)^2} = \frac{g'(\xi)}{p-q} = \frac{(\xi-q)T''(\xi)}{p-q} = (1-\theta)T''(\xi).$$

By Lemma 3.3, $T''(\xi) = -xyI_1(\ln b - \ln a)^2$, $x = a^\xi, y = b^\xi$. Obviously, when $I_1 < (>)0$, we get $\frac{\partial\ln\mathcal{H}_f}{\partial p} > (<)0$.

2) When $p = q$, from (2.2) and (3.6),

$$\ln \mathcal{H}_f = \ln G_f^{\frac{1}{p}}(a^p, b^p) = \frac{x f_x(x,y)\ln x + y f_y(x,y)\ln y}{f(x,y)} = T'(p),$$

$$\frac{\partial \ln \mathcal{H}_f}{\partial p} = T''(p) = -xyI_1(\ln b - \ln a)^2.$$

when $I_1 < (>)0$, we get $\frac{\partial \ln \mathcal{H}_f}{\partial p} > (<)0$.

Combining 1) with 2), the proof is completed. □

Proof of Corollary 2.4. It follows from Theorem 2.3 that the monotonicity of $\mathcal{H}_f(p,q)$ depends on the sign of $I_1 = (\ln f)_{xy}$.

1) For $f(x,y) = L(x,y)$,

$$\begin{aligned} I_1 = (\ln f)_{xy} &= \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2} \\ &= \frac{1}{xy(x-y)^2}\left((\sqrt{xy})^2 - L^2(x,y)\right). \end{aligned}$$

By the well-known inequality $L(x,y) > \sqrt{xy}$ ([13]), we have $I_1 < 0$.

2) For $f(x,y) = A(x,y)$,

$$I_1 = (\ln f)_{xy} = -\frac{1}{(x+y)^2} < 0.$$

3) For $f(x,y) = E(x,y)$,

$$\begin{aligned} I_1 = (\ln f)_{xy} &= \frac{1}{(x-y)^3}\left[2(x-y) - (x+y)(\ln x - \ln y)\right] \\ &= \frac{2(\ln x - \ln y)}{(x-y)^3}\left[L(x,y) - \frac{x+y}{2}\right]. \end{aligned}$$

By the well-known inequality $L(x,y) < \frac{x+y}{2}$ ([13]), we have $I_1 < 0$.

4) For $f(x,y) = D(x,y)$,

$$I_1 = (\ln f)_{xy} = \frac{1}{(x-y)^2} > 0.$$

Applying mechanically Theorem 2.3, we immediately obtain Corollary 2.4. □

Proof of Theorem 2.5.

1) Since

$$\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{1}{p-q}\left[\frac{pa^{p-1} f_x(a^p, b^p)}{f(a^p, b^p)} - \frac{qa^{q-1} f_x(a^q, b^q)}{f(a^q, b^q)}\right] = \frac{S(p) - S(q)}{a(p-q)},$$

by the Mean-value Theorem, there exists $\xi = q + \theta(p-q)$ with $\theta \in (0,1)$, such that

$$\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{S(p) - S(q)}{a(p-q)} = a^{-1}S'(\xi).$$

From Lemma 3.4, $S'(\xi) = xyI_{2a}$, where $x = a^\xi, y = b^\xi$. Obviously ,if $I_{2a} > 0$, then $\frac{\partial \ln \mathcal{H}_f}{\partial a} > 0$, so $\mathcal{H}_f(a,b)$ is strictly increasing in a; If $I_{2a} < 0$, then $\frac{\partial \ln \mathcal{H}_f}{\partial a} < 0$, so $\mathcal{H}_f(a,b)$ is strictly decreasing in a.

2) It can be proved in the same way. □

Proof of Corollary 2.6.
1) For $f(x,y)=L(x,y)$,

$$I_{2a}=\left[(\ln f)_x \ln(y/x)\right]_y=\frac{x/y-1-\ln(x/y)}{(x-y)^2}.$$

By the well-known inequality $\ln x < x-1$ $(x>0, x\neq 1)$, we have $I_{2a}>0$.
2) For $f(x,y)=D(x,y)$,

$$I_{2a}=\left[(\ln f)_x \ln(y/x)\right]_y=\frac{x/y-1-\ln(x/y)}{(x-y)^2}>0.$$

Since $\mathcal{H}_L(a,b), \mathcal{H}_D(a,b)$ are both symmetric with respect to a and b, applying mechanically Theorem 2.5, we immediately obtain Corollary 2.6. $\square$

4. Some Applications

As direct applications of theorems and lemmas in this paper, we will present several examples as follows.

Example 4.1 (a G-A inequality chain). By 1) of Corollary 2.4, for $f(x,y)=A(x,y), L(x,y)$ and $E(x,y)$, $\mathcal{H}_f(p,q)$ are strictly increasing in both p and q. So there are

$$\begin{aligned}\mathcal{H}_f(a,b;0,0)<\mathcal{H}_f(a,b;1,0)<\mathcal{H}_f\left(a,b;1,\frac{1}{2}\right)\\ <\mathcal{H}_f(a,b;1,1)<\mathcal{H}_f(a,b;1,2).\end{aligned}\tag{4.1}$$

From it we can obtain the following inequalities respectively, that are

$$\sqrt{ab}<L(a,b)<\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2<E(a,b)<\frac{a+b}{2};\tag{4.2}$$

$$\sqrt{ab}<\frac{a+b}{2}<\left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)^2<Z(a,b)<\frac{a^2+b^2}{a+b};\tag{4.3}$$

$$\sqrt{ab}<E(a,b)<\left[\frac{E(a,b)}{E\left(\sqrt{a},\sqrt{b}\right)}\right]^2<Y(a,b)<\frac{E(a^2,b^2)}{E(a,b)}.\tag{4.4}$$

Notice $\frac{E(a^2,b^2)}{E(a,b)}=Z(a,b)$, then (4.4) can be written into that

$$\sqrt{ab}<E(a,b)<Z^2\left(\sqrt{a},\sqrt{b}\right)<E\exp\left(1-\frac{G^2}{L^2}\right)<Z(a,b).\tag{4.5}$$

The inequality (4.2) was proved by [13], which shows that can be inserted L, $\frac{A+G}{2}$ and E between G and A, so we call (4.2) the G-A inequality chain. (4.3) and (4.4) are the same in form completely, so we call (4.1) the G-A inequality chain for homogeneous functions.

Remark 4.1. That $\frac{E(a^2,b^2)}{E(a,b)} = Z(a,b)$ is a new identical equation for mean. In fact,

$$\begin{aligned} E(a,b)Z(a,b) &= e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} b^{\frac{b}{b+a}} a^{\frac{a}{b+a}} \\ &= e^{-1} b^{\frac{b}{b+a}+\frac{b}{b-a}} a^{\frac{a}{b+a}-\frac{a}{b-a}} \\ &= e^{-1} b^{\frac{2b^2}{b^2-a^2}} a^{\frac{-2a^2}{b^2-a^2}} \\ &= e^{-1}\left(\frac{(b^2)^{b^2}}{(a^2)^{a^2}}\right)^{\frac{1}{b^2-a^2}} = E(a^2,b^2). \end{aligned}$$

It shows that $Z(a,b)$ is not only one "geometric mean", but also one ratio of one exponential mean to another. Thus inequalities involving $Z(a,b)$ may be translated into inequalities involving exponential mean.

Example 4.2 (**An estimation for upper bound of Stolarsky mean).** From 2) of Corollary 2.4, we can prove expediently an estimation for the upper bound of the Stolarsky mean presented by [12]:

$$S_p(a,b) < p^{\frac{1}{1-p}}(a+b) \text{ with } p > 2, \text{ where } S_p(a,b) = \left(\frac{b^p - a^p}{p(b-a)}\right)^{\frac{1}{p-1}}.$$

In fact, from 2) of Corollary 2.4, when $p,q \in (-\infty,0) \cup (0,+\infty)$, $\mathcal{H}_D(p,q)$ is strictly decreasing in both p and q, so when $p > 2$, we have $\mathcal{H}_D(a,b;1,p) < \mathcal{H}_D(a,b;1,2)$.

Notice

$$\mathcal{H}_D(a,b;p,1) = \left(\frac{a^p - b^p}{a-b}\right)^{\frac{1}{p-1}} = p^{\frac{1}{p-1}} S_p(a,b) \quad (p > 0), \tag{4.6}$$

thus when $p > 2$, we obtain $p^{\frac{1}{p-1}} S_p(a,b) < 2^{\frac{1}{2-1}} S_2(a,b) = a+b$, i.e. $S_p(a,b) < p^{\frac{1}{1-p}}(a+b)$.

Example 4.3 (**Reversed inequalities and estimations for exponential mean).** By 1) of Corollary 2.4, $\mathcal{H}_L(p,q)$ is strictly increasing in both p and q, so when $p_1 \in (0,1)$, $p_2 \in (1,+\infty)$, we have

$$\mathcal{H}_L(a,b;p_1,1) < \mathcal{H}_L(a,b;1,1) < \mathcal{H}_L(a,b;p_2,1),$$

i.e.

$$S_{p_1}(a,b) < E(a,b) < S_{p_2}(a,b). \tag{4.7}$$

On the other hand, By 2) of Corollary 2.4, when $p,q \in (-\infty,0) \cup (0,+\infty)$, $\mathcal{H}_D(p,q)$ is strictly monotone decreasing in both p and q. So when $p_1 \in (0,1)$, $p_2 \in (1,+\infty)$, we have

$$\mathcal{H}_D(a,b;p_1,1) > \mathcal{H}_D(a,b;1,1) > \mathcal{H}_D(a,b;p_2,1). \tag{4.8}$$

From (4.6), (4.8) can be written into

$$p_2^{\frac{1}{p_2-1}} S_{p_2}(a,b) < eE(a,b) < p_1^{\frac{1}{p_1-1}} S_{p_1}(a,b)$$

or

$$\frac{1}{e} p_2^{\frac{1}{p_2-1}} S_{p_2}(a,b) < E(a,b) < \frac{1}{e} p_1^{\frac{1}{p_1-1}} S_{p_1}(a,b). \tag{4.9}$$

Combining (4.7) with (4.9), we have

$$S_{p_1}(a,b) < E(a,b) < \frac{1}{e} p_1^{\frac{1}{p_1-1}} S_{p_1}(a,b), \text{ where } p_1 \in (0,1), \tag{4.10}$$

$$\frac{1}{e} p_2^{\frac{1}{p_2-1}} S_{p_2}(a,b) < E(a,b) < S_{p_2}(a,b), \text{ where } p_2 \in (1,+\infty). \tag{4.11}$$

In particular, when $p_1 = \frac{1}{2}, p_2 = 2$, by (4.10), (4.11), we get

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 < E(a,b) < \frac{4}{e}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2, \tag{4.12}$$

$$\frac{2}{e}\left(\frac{a+b}{2}\right) < E(a,b) < \frac{a+b}{2}. \tag{4.13}$$

The inequalities (4.12) and (4.13) may be denoted simply by

$$\frac{A+G}{2} < E < \frac{4}{e}\frac{A+G}{2}, \tag{4.14}$$

$$\frac{2}{e}A < E < A. \tag{4.15}$$

The inequalities (4.14) and (4.15) make certain a bound of error that exponential mean E are estimated by A or $\frac{A+G}{2}$.

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