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ON A CONJECTURE OF QI-TYPE INTEGRAL INEQUALITIES

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ABSTRACT. A conjecture by Chen and Kimball on Qi-type integral inequalities is proven to be true.

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Recently, Chen and Kimball [1], studied a very interesting Qi-type integral inequality and proved the following result.

Theorem 1. *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has a derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a)=0$ for $i=0,1,2,\dots,n-1$. If $f^{(n)}(x)\geq \frac{n!}{(n+1)^{(n-1)}}$ and $f^{(n)}(x)$ is increasing, then*

$$\int_a^b [f(x)]^{n+2}dx \geq \left[\int_a^b f(x)dx\right]^{n+1}. \tag{1}$$

If $0\leq f^{(n)}(x)\leq \frac{n!}{(n+1)^{(n-1)}}$ and $f^{(n)}(x)$ is decreasing, then the inequality (1) is reversed.

In this theorem and in the sequel, $f^{(0)}(a)$ stands for $f(a)$.

In [1], Chen and Kimball conjectured that the additional hypothesis on monotonicity in Theorem 1 could be dropped:

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Theorem 2 (Conjecture). *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has derivative of the n-th order on the interval $[a, b]$ such that $f^{(i)}(a) = 0$ for $i = 0, 1, 2, \ldots, n-1$.*

(i) *If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$, then the inequality (1) holds.*
(ii) *If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$, then the inequality (1) is reversed.*

In this article, we prove by mathematical induction that the conjecture is true. As a matter of fact, Theorem 2 holds under slightly weaker assumptions (existence of $f^{(n)}(x)$ at the endpoints $x = a, x = b$ is not needed). We start by applying Cauchy's mean value theorem (CMVT) (that is, the statement that for f, g differentiable on (a, b) and continuous on $[a, b]$ there exists a $\xi \in (a, b)$ such that

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)))$$

to prove the following lemma, which will in turn be used to prove Theorem 2.

Lemma 3. *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has a derivative of the n-th order on the interval (a, b) and $f^{(n-1)}(x)$ is continuous on $[a, b]$ such that $f^{(i)}(a) = 0$ for $i = 0, 1, 2, \ldots, n-1$.*

(i) *If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$, then*

$$(f(x))^{n+1} \geq (n+1) \left(\int_a^x f(s)ds \right)^n \quad \textit{for} \quad x \in [a, b].$$

(ii) *If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$, then*

$$(f(x))^{n+1} \leq (n+1) \left(\int_a^x f(s)ds \right)^n \quad \textit{for} \quad x \in [a, b].$$

Proof. First notice that if f is identically 0, then the statement is trivially true. Suppose that f is not identically 0 on $[a, b]$. Then the assumption implies that $f(x) \geq 0$ for $x \in [a, b]$. If $\int_a^x f(s)ds = 0$ for some $x \in (a, b]$ then $f(s) = 0$ for all $s \in [a, x]$. So we can assume that $\int_a^x f(s)ds > 0$ for all $x \in (a, b]$. Otherwise, we can find $a_1 \in (a, b)$ such that $\int_a^x f(s)ds = 0$ for $x \in [a, a_1]$ and $\int_a^x f(s)ds > 0$ for $x \in (a_1, b)$ and hence we only need to consider f on $[a_1, b]$.
(i) Suppose that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$.

(1) $n = 1$. By CMVT, for every $x \in (a, b]$, there exists a $b_1 \in (a, x)$ such that

$$\frac{(f(x))^2}{2\int_a^x f(s)ds} = \frac{2f(b_1)f'(b_1)}{2f(b_1)} = f'(b_1) \geq 1.$$

So (i) is true for $n = 1$.

(2) Suppose that (i) is true for $n = k > 1$. We prove that (i) is true for $n = k + 1$. It then follows by mathematical induction that (i) is true for $n = 1, 2, \ldots$.

By CMVT, for every $x \in (a, b]$, there exists a $b_1 \in (a, x)$ such that

$$\begin{aligned}
\frac{(f(x))^{k+2}}{(k+2)\left(\int_a^x f(s)ds\right)^{k+1}} &= \frac{1}{(k+2)}\left(\frac{(f(x))^{\frac{k+2}{k+1}}}{\int_a^x f(s)ds}\right)^{k+1} \\
&= \frac{1}{(k+2)}\left(\frac{\frac{k+2}{k+1}(f(b_1))^{\frac{1}{k+1}} f'(b_1)}{f(b_1)}\right)^{k+1} \\
&= \frac{(k+2)^k}{(k+1)^{k+1}}\frac{(f'(b_1))^{k+1}}{(f(b_1)^k} \\
&= \frac{\left(\left(\frac{k+2}{k+1}\right)^k f'(b_1)\right)^{k+1}}{(k+1)\left(\int_a^{b_1}\left(\frac{k+2}{k+1}\right)^k f'(s)ds\right)^k} \geq 1.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{d^k}{dx^k}\left[\left(\frac{k+2}{k+1}\right)^k f'(x)\right] &= \left(\frac{k+2}{k+1}\right)^k f^{(k+1)}(x) \\
&\geq \left(\frac{k+2}{k+1}\right)^k \frac{(k+1)!}{(k+2)^k} \\
&= \frac{k!}{(k+1)^{k-1}}
\end{aligned}$$

for $x \in (a, b)$, by the induction assumption that (i) is true for $n = k$.

So (i) is true for $n = 1, 2, \ldots$.

(ii) The proof of the second part is similar so we leave out the details. This completes the proof of the lemma. □

Now we are in a position to prove the conjecture (Theorem 2).

Proof of Conjecture (Theorem 2). As in the proof of Lemma 3, we can assume that $\int_a^x f(s)ds > 0$ for any $x \in (a, b]$.
(i) Suppose that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$. By CMVT and Lemma 3, in case (i), there exists a $b_1 \in (a, x)$ such that

$$\frac{\int_a^b [f(x)]^{n+2}dx}{\left[\int_a^b f(x)dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)dx\right]^n} \geq 1.$$

This proves (i).
(ii) Suppose that $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a, b)$.
By CMVT and Lemma 3, in case (ii), there exists a $b_1 \in (a, x)$ such that

$$\frac{\int_a^b [f(x)]^{n+2}dx}{\left[\int_a^b f(x)dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)dx\right]^n} \leq 1.$$

This completes the proof of the conjecture. □

As the proofs show, we actually have the following slightly stronger result which is a generalization of Proposition 1.1 in [2] and Theorem 4 and Theorem 5 in [1].

Theorem 4. *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has derivative of the n-th order on the interval (a,b) and $f^{(n-1)}(x)$ is continuous on $[a,b]$ such that $f^{(i)}(a)=0$ for $i=0,1,2,\ldots,n-1$.*

(i) *If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then the inequality (1) holds.*

(ii) *If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then the inequality (1) is reversed.*

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