

THREE MAPPINGS RELATED TO CHEBYSHEV-TYPE INEQUALITIES

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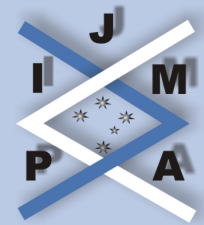
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Key words: Chebyshev-type inequality, Monotonicity, Refinement.

Abstract: In this paper, by the Chebyshev-type inequalities we define three mappings, investigate their main properties, give some refinements for Chebyshev-type inequalities, obtain some applications.

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**Mappings Related to
Chebyshev-type Inequalities**
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1. Introduction

Let $n(\geq 2)$ be a given positive integer, $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ be known as sequences of real numbers. Also, let $p_i > 0$ and $q_i > 0$ ($i = 1, 2, \dots, n$), $P_j = p_1 + p_2 + \dots + p_j$ and $Q_j = q_1 + q_2 + \dots + q_j$ ($j = 1, 2, \dots, n$).

If A and B are both increasing or both decreasing, then

$$(1.1) \quad \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right) \leq n \sum_{i=1}^n a_i b_i.$$

If one of the sequences A or B is increasing and the other decreasing, then the inequality (1.1) is reversed.

The inequality (1.1) is called the Chebyshev's inequality, see [1, 2].

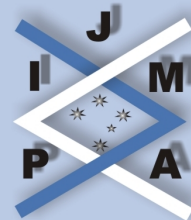
For A and B both increasing or both decreasing, Behdjet in [3] extended inequality (1.1) to

$$(1.2) \quad \left(\sum_{i=1}^n p_i a_i \right) \left(\sum_{i=1}^n q_i b_i \right) + \left(\sum_{i=1}^n q_i a_i \right) \left(\sum_{i=1}^n p_i b_i \right) \\ \leq P_n \sum_{i=1}^n q_i a_i b_i + Q_n \sum_{i=1}^n p_i a_i b_i.$$

If one of the sequences A or B is increasing and the other decreasing, then the inequality (1.2) is reversed.

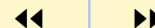
For $p_i = q_i$, $i = 1, 2, \dots, n$, the inequality (1.2) reduces to

$$(1.3) \quad \left(\sum_{i=1}^n p_i a_i \right) \left(\sum_{i=1}^n p_i b_i \right) \leq P_n \sum_{i=1}^n p_i a_i b_i,$$



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where, A and B are both increasing or both decreasing. If one of the sequences A or B is increasing and the other decreasing, then the inequality (1.3) is reversed.

Let $r, s : [a, b] \rightarrow \mathbb{R}$ be integrable functions, either both increasing or both decreasing. Furthermore, let $p, q : [a, b] \rightarrow [0, +\infty)$ be the integrable functions. Then

$$(1.4) \quad \int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt \\ \leq \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt.$$

If one of the functions r or s is increasing and the other decreasing, then the inequality (1.4) is reversed.

When $p(t) = q(t)$, $t \in [a, b]$, the inequality (1.4) reduces to

$$(1.5) \quad \int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt,$$

where r and s are both increasing or both decreasing. If one of the functions r or s is increasing and the other decreasing, then the inequality (1.5) is reversed.

Inequalities (1.4) and (1.5) are the integral forms of inequalities (1.2) and (1.3), respectively (see [1, 2]).

The results from other inequalities connected with (1.1) to (1.5) can be seen in [1], [3] – [8] and [2, pp. 61–65].

We define three mappings c , C and \tilde{C} by $c : \mathbb{N}_+ \times \mathbb{N}_+ \rightarrow \mathbb{R}$,

$$(1.6) \quad c(k, n; p_i, q_i) = P_k \sum_{i=1}^k q_i a_i b_i + Q_k \sum_{i=1}^k p_i a_i b_i$$

$$\begin{aligned}
& + \left(\sum_{i=k+1}^n p_i a_i \right) \left(\sum_{i=1}^n q_i b_i \right) + \left(\sum_{i=1}^k p_i a_i \right) \left(\sum_{i=k+1}^n q_i b_i \right) \\
& + \left(\sum_{i=k+1}^n q_i a_i \right) \left(\sum_{i=1}^n p_i b_i \right) + \left(\sum_{i=1}^k q_i a_i \right) \left(\sum_{i=k+1}^n p_i b_i \right),
\end{aligned}$$

where $k = 1, 2, \dots, n$, and

$$\sum_{i=n+1}^n q_i a_i = \sum_{i=n+1}^n p_i b_i = \sum_{i=n+1}^n p_i a_i = \sum_{i=n+1}^n q_i b_i = 0$$

is assumed.

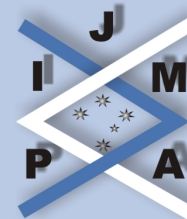
For $C : [a, b] \rightarrow \mathbb{R}$,

(1.7) $C(x; p, q; r, s)$

$$\begin{aligned}
& = \int_a^x p(t) dt \int_a^x q(t) r(t) s(t) dt + \int_a^x q(t) dt \int_a^x p(t) r(t) s(t) dt \\
& + \int_x^b p(t) r(t) dt \int_a^b q(t) s(t) dt + \int_a^x p(t) r(t) dt \int_x^b q(t) s(t) dt \\
& + \int_x^b q(t) r(t) dt \int_a^b p(t) s(t) dt + \int_a^x q(t) r(t) dt \int_x^b p(t) s(t) dt
\end{aligned}$$

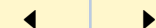
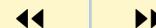
and for $\tilde{C} : [a, b] \rightarrow \mathbb{R}$,

$$(1.8) \quad \tilde{C}(y; p, q; r, s) = \int_y^b p(t) dt \int_y^b q(t) r(t) s(t) dt + \int_y^b q(t) dt \int_y^b p(t) r(t) s(t) dt$$



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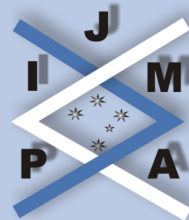


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$$\begin{aligned}
 &+ \int_a^y p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_y^b p(t)r(t)dt \int_a^y q(t)s(t)dt \\
 &\quad + \int_a^y q(t)r(t)dt \int_a^b p(t)s(t)dt + \int_y^b q(t)r(t)dt \int_a^y p(t)s(t)dt.
 \end{aligned}$$

We write

$$\begin{aligned}
 (1.9) \quad c_1(k, n; p_i) &= \frac{1}{2}c(k, n; p_i, p_i) \\
 &= P_k \sum_{i=1}^k p_i a_i b_i + \left(\sum_{i=k+1}^n p_i a_i \right) \left(\sum_{i=1}^n p_i b_i \right) + \left(\sum_{i=1}^k p_i a_i \right) \left(\sum_{i=k+1}^n p_i b_i \right),
 \end{aligned}$$

$$\begin{aligned}
 (1.10) \quad c_2(k, n) &= c_1(k, n; 1) \\
 &= k \sum_{i=1}^k a_i b_i + \left(\sum_{i=k+1}^n a_i \right) \left(\sum_{i=1}^n b_i \right) + \left(\sum_{i=1}^k a_i \right) \left(\sum_{i=k+1}^n b_i \right),
 \end{aligned}$$

$$\begin{aligned}
 (1.11) \quad C_0(x; p; r, s) &= \frac{1}{2}C(x; p, p; r, s) \\
 &= \int_a^x p(t)dt \int_a^x p(t)r(t)s(t)dt + \int_x^b p(t)r(t)dt \int_a^b p(t)s(t)dt \\
 &\quad + \int_a^x p(t)r(t)dt \int_x^b p(t)s(t)dt
 \end{aligned}$$

and

$$\begin{aligned}(1.12) \quad & \tilde{C}_0(y; p; r, s) \\ &= \frac{1}{2} \tilde{C}(y; p, p; r, s) \\ &= \int_y^b p(t) dt \int_y^b p(t) r(t) s(t) dt + \int_a^y p(t) r(t) dt \int_a^b p(t) s(t) dt \\ &\quad + \int_y^b p(t) r(t) dt \int_a^y p(t) s(t) dt.\end{aligned}$$

(1.10), (1.6), (1.9), (1.7) and (1.8), (1.11) and (1.12) are generated by the inequalities (1.1) to (1.5), respectively.

The aim of this paper is to study the monotonicity properties of c , C and \tilde{C} , and obtain some refinements of (1.1) to (1.5) using these monotonicity properties. Some applications are given.



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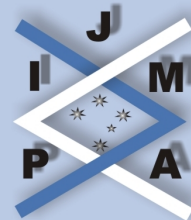
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2. Main Results

The monotonicity properties of the mapping c , c_1 and c_2 are embodied in the following theorem.

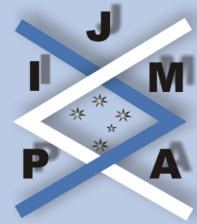
Theorem 2.1. *Let c , c_1 and c_2 be defined as in the first section. If A and B are both increasing or both decreasing, then we have the following refinements of (1.2), (1.3) and (1.1)*

$$(2.1) \quad \left(\sum_{i=1}^n p_i a_i \right) \left(\sum_{i=1}^n q_i b_i \right) + \left(\sum_{i=1}^n q_i a_i \right) \left(\sum_{i=1}^n p_i b_i \right) \\ = c(1, n; p_i, q_i) \leq \cdots \leq c(k, n; p_i, q_i) \leq c(k+1, n; p_i, q_i) \leq \cdots \\ \leq c(n, n; p_i, q_i) = P_n \sum_{i=1}^n q_i a_i b_i + Q_n \sum_{i=1}^n p_i a_i b_i,$$

$$(2.2) \quad \left(\sum_{i=1}^n p_i a_i \right) \left(\sum_{i=1}^n p_i b_i \right) = c_1(1, n; p_i) \leq \cdots \leq c_1(k, n; p_i) \\ \leq c_1(k+1, n; p_i) \leq \cdots \leq c_1(n, n; p_i) \\ = P_n \sum_{i=1}^n p_i a_i b_i$$

and

$$(2.3) \quad \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right) = c_2(1, n) \leq \cdots \leq c_2(k, n) \\ \leq c_2(k+1, n) \leq \cdots \leq c_2(n, n) = n \sum_{i=1}^n a_i b_i,$$



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respectively. If one of the sequences A or B is increasing and the other decreasing, then inequalities in (2.1)–(2.3) are reversed.

The monotonicity properties of the mappings C and C_0 are given in the following theorem.

Theorem 2.2. *Let C and C_0 be defined as in the first section. If r and s are both increasing or both decreasing, then $C(x; p, q; r, s)$ and $C_0(x; p; r, s)$ are increasing on $[a, b]$ with x , and for $x \in [a, b]$ we have the following refinements of (1.4) and (1.5)*

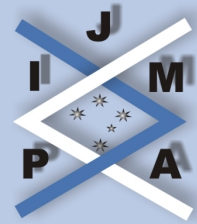
$$\begin{aligned}
 (2.4) \quad & \int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt \\
 & = C(a; p, q; r, s) \leq C(x; p, q; r, s) \leq C(b; p, q; r, s) \\
 & = \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt
 \end{aligned}$$

and

$$\begin{aligned}
 (2.5) \quad & \int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt = C_0(a; p; r, s) \\
 & \leq C_0(x; p; r, s) \leq C_0(b; p; r, s) \\
 & = \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt,
 \end{aligned}$$

respectively. If one of the functions r or s is increasing and the other decreasing, then $C(x; p, q; r, s)$ and $C_0(x; p; r, s)$ are decreasing on $[a, b]$ with x , and inequalities in (2.4) and (2.5) are reversed.

The monotonicity properties of \tilde{C} and \tilde{C}_0 are given in the following theorem.



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Theorem 2.3. Let \tilde{C} and \tilde{C}_0 be defined as in the first section. If r and s are both increasing or both decreasing, then $\tilde{C}(y; p, q; r, s)$ and $\tilde{C}_0(y; p; r, s)$ are decreasing on $[a, b]$ with y , and for $y \in [a, b]$ we have the following refinements of (1.4) and (1.5)

$$\begin{aligned}(2.6) \quad & \int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt \\ & = \tilde{C}(b; p, q; r, s) \leq \tilde{C}(y; p, q; r, s) \leq \tilde{C}(a; p, q; r, s) \\ & = \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt\end{aligned}$$

and

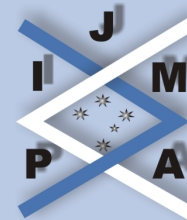
$$\begin{aligned}(2.7) \quad & \int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt = \tilde{C}_0(b; p; r, s) \leq \tilde{C}_0(y; p; r, s) \leq \tilde{C}_0(a; p; r, s) \\ & = \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt,\end{aligned}$$

respectively. If one of the functions r or s is increasing and the other decreasing, then $\tilde{C}(y; p, q; r, s)$ and $\tilde{C}_0(y; p; r, s)$ are increasing on $[a, b]$ with y , and the inequalities in (2.6) and (2.7) are reversed.

3. Proof of Theorems

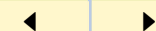
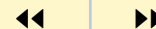
Proof of Theorem 2.1. For $k = 2, 3, \dots, n$, we have

$$\begin{aligned}
 (3.1) \quad & c(k, n; p_i, q_i) - c(k-1, n; p_i, q_i) \\
 &= (P_{k-1} + p_k) \left(\sum_{i=1}^{k-1} q_i a_i b_i + q_k a_k b_k \right) \\
 &\quad + (Q_{k-1} + q_k) \left(\sum_{i=1}^{k-1} p_i a_i b_i + p_k a_k b_k \right) \\
 &\quad - \left[P_{k-1} \sum_{i=1}^{k-1} q_i a_i b_i + Q_{k-1} \sum_{i=1}^{k-1} p_i a_i b_i \right] + \sum_{i=k+1}^n p_i a_i \sum_{i=1}^n q_i b_i \\
 &\quad + \left(p_k a_k + \sum_{i=1}^{k-1} p_i a_i \right) \sum_{i=k+1}^n q_i b_i - \left(p_k a_k + \sum_{i=k+1}^n p_i a_i \right) \sum_{i=1}^n q_i b_i \\
 &\quad - \sum_{i=1}^{k-1} p_i a_i \left(q_k b_k + \sum_{i=k+1}^n q_i b_i \right) + \sum_{i=k+1}^n q_i a_i \sum_{i=1}^n p_i b_i \\
 &\quad + \left(q_k a_k + \sum_{i=1}^{k-1} q_i a_i \right) \sum_{i=k+1}^n p_i b_i - \left(q_k a_k + \sum_{i=k+1}^n q_i a_i \right) \sum_{i=1}^n p_i b_i \\
 &\quad - \sum_{i=1}^{k-1} q_i a_i \left(p_k b_k + \sum_{i=k+1}^n p_i b_i \right) \\
 &= \left[p_k \sum_{i=1}^{k-1} q_i a_i b_i + p_k a_k b_k \sum_{i=1}^{k-1} q_i - p_k a_k \sum_{i=1}^{k-1} q_i b_i - p_k b_k \sum_{i=1}^{k-1} q_i a_i \right]
 \end{aligned}$$



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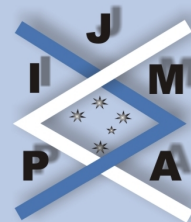


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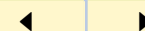
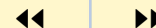
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$$\begin{aligned}
 & + \left[q_k \sum_{i=1}^{k-1} p_i a_i b_i + q_k a_k b_k \sum_{i=1}^{k-1} p_i - q_k a_k \sum_{i=1}^{k-1} p_i b_i - q_k b_k \sum_{i=1}^{k-1} p_i a_i \right] \\
 & = p_k \sum_{i=1}^{k-1} q_i (a_k - a_i) (b_k - b_i) + q_k \sum_{i=1}^{k-1} p_i (a_k - a_i) (b_k - b_i).
 \end{aligned}$$

If A and B are both increasing or both decreasing, then

$$(3.2) \quad (a_k - a_i)(b_k - b_i) \geq 0, \quad (i = 1, 2, \dots, k-1).$$

Using (1.6), (3.1) and (3.2), we obtain (2.1).

If one of the sequences A or B is increasing and the other decreasing, then (3.2) is reversed, which implies that the inequalities in (2.1) are reversed.

For $i = 1, 2, \dots, n$, replacing q_i in (2.1) with p_i and replacing p_i in (2.2) with 1, we obtain (2.2) and (2.3), respectively. This completes the proof of Theorem 2.1. \square

Proof of Theorem 2.2. For any $x_1, x_2 \in [a, b]$, $x_1 < x_2$, we write

$$\begin{aligned}
 I_1 = & \int_{x_1}^{x_2} p(t) dt \int_{x_1}^{x_2} q(t) r(t) s(t) dt + \int_{x_1}^{x_2} q(t) dt \int_{x_1}^{x_2} p(t) r(t) s(t) dt \\
 & - \int_{x_1}^{x_2} p(t) r(t) dt \int_{x_1}^{x_2} q(t) s(t) dt - \int_{x_1}^{x_2} q(t) r(t) dt \int_{x_1}^{x_2} p(t) s(t) dt.
 \end{aligned}$$

For $t \in [a, x_1]$, $u \in [x_1, x_2]$, using the properties of double integrals, we get

$$I_2 = \iint_{[a, x_1] \times [x_1, x_2]} p(t) q(u) (r(t) - r(u)) (s(t) - s(u)) dt du$$

$$\begin{aligned}
&= \int_a^{x_1} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_a^{x_1} p(t)r(t)s(t)dt \int_{x_1}^{x_2} q(t)dt \\
&\quad - \int_a^{x_1} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_a^{x_1} p(t)s(t)dt \int_{x_1}^{x_2} q(t)r(t)dt
\end{aligned}$$

and

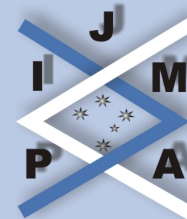
$$\begin{aligned}
I_3 &= \iint_{[a,x_1] \times [x_1,x_2]} p(u)q(t) \left(r(t) - r(u) \right) \left(s(t) - s(u) \right) dt du \\
&= \int_a^{x_1} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt + \int_a^{x_1} q(t)r(t)s(t)dt \int_{x_1}^{x_2} p(t)dt \\
&\quad - \int_a^{x_1} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt - \int_a^{x_1} q(t)s(t)dt \int_{x_1}^{x_2} p(t)r(t)dt.
\end{aligned}$$

When $x_1 = a$, from (1.7), we get

$$\begin{aligned}
(3.3) \quad &C(x_2; p, q, r, s) - C(x_1; p, q, r, s) \\
&= \int_{x_1}^{x_2} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_{x_1}^{x_2} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt \\
&\quad - \int_{x_1}^{x_2} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_{x_1}^{x_2} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt \\
&= I_1.
\end{aligned}$$

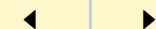
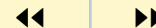
When $x_1 > a$, from (1.7), we have

$$(3.4) \quad C(x_2; p, q, r, s) - C(x_1; p, q, r, s)$$



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$$\begin{aligned}
&= \left(\int_a^{x_1} + \int_{x_1}^{x_2} \right) p(t) dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} \right) q(t) r(t) s(t) dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2} \right) q(t) dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} \right) p(t) r(t) s(t) dt \\
&\quad - \int_a^{x_1} p(t) dt \int_a^{x_1} q(t) r(t) s(t) dt - \int_a^{x_1} q(t) dt \int_a^{x_1} p(t) r(t) s(t) dt \\
&\quad + \int_{x_2}^b p(t) r(t) dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^b \right) q(t) s(t) dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2} \right) p(t) r(t) dt \int_{x_2}^b q(t) s(t) dt \\
&\quad - \left(\int_{x_1}^{x_2} + \int_{x_2}^b \right) p(t) r(t) dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^b \right) q(t) s(t) dt \\
&\quad - \int_a^{x_1} p(t) r(t) dt \left(\int_{x_1}^{x_2} + \int_{x_2}^b \right) q(t) s(t) dt \\
&\quad + \int_{x_2}^b q(t) r(t) dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^b \right) p(t) s(t) dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2} \right) q(t) r(t) dt \int_{x_2}^b p(t) s(t) dt \\
&\quad - \left(\int_{x_1}^{x_2} + \int_{x_2}^b \right) q(t) r(t) dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^b \right) p(t) s(t) dt \\
&\quad - \int_a^{x_1} q(t) r(t) dt \left(\int_{x_1}^{x_2} + \int_{x_2}^b \right) p(t) s(t) dt
\end{aligned}$$

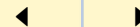
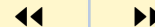


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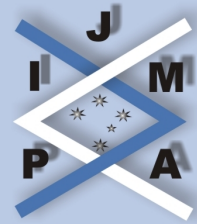
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$$\begin{aligned}
 &= \left[\int_{x_1}^{x_2} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_{x_1}^{x_2} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt \right. \\
 &\quad \left. - \int_{x_1}^{x_2} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_{x_1}^{x_2} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt \right] \\
 &\quad + \left[\int_a^{x_1} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_a^{x_1} p(t)r(t)s(t)dt \int_{x_1}^{x_2} q(t)dt \right. \\
 &\quad \left. - \int_a^{x_1} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_a^{x_1} p(t)s(t)dt \int_{x_1}^{x_2} q(t)r(t)dt \right] \\
 &\quad + \left[\int_a^{x_1} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt + \int_a^{x_1} q(t)r(t)s(t)dt \int_{x_1}^{x_2} p(t)dt \right. \\
 &\quad \left. - \int_a^{x_1} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt - \int_a^{x_1} q(t)s(t)dt \int_{x_1}^{x_2} p(t)r(t)dt \right] \\
 &= I_1 + I_2 + I_3.
 \end{aligned}$$

(1) If r and s are both increasing or both decreasing, then we have

$$(3.5) \quad (r(t) - r(u))(s(t) - s(u)) \geq 0,$$

i.e., $I_2 \geq 0$ and $I_3 \geq 0$. By the inequality (1.4), $I_1 \geq 0$ holds. Using (3.3) and (3.4), we obtain that $C(x; p, q; r, s)$ is increasing on $[a, b]$ with x . Further, from (1.11), we get that $C_0(x; p; r, s)$ is increasing on $[a, b]$ with x .

From (1.7) and (1.11), using the increasing properties of $C(x; p, q; r, s)$ and $C_0(x; p; r, s)$, we obtain (2.4) and (2.5), respectively.

(2) If one of the functions r or s is increasing and the other decreasing, then the inequality in (3.5) is reversed, which implies that $I_2 \leq 0$ and $I_3 \leq 0$. By the reverse of (1.4), $I_1 \leq 0$ holds. From (3.3) and (3.4), (1.11), we obtain that $C(x; p, q; r, s)$, $C_0(x; p; r, s)$ are decreasing on $[a, b]$ with x , respectively.

From (1.7) and (1.11), using the decreasing properties of $C(x; p, q; r, s)$ and $C_0(x; p; r, s)$, we obtain the reverse of (2.4) and (2.5), respectively.

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Using the same arguments as those in the proof of Theorem 2.2, we can prove Theorem 2.3. □



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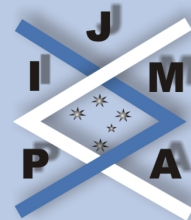
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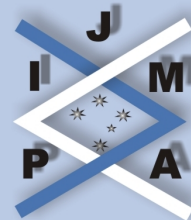
4. Applications

Let \mathbf{I} be a real interval and $u, v, w : \mathbf{I} \rightarrow [0, +\infty)$. For any $\alpha, \beta \in \mathbb{R}$ and any $x_i \in \mathbf{I}$ ($i = 1, 2, \dots, n, n \geq 2$) satisfying $x_1 \leq x_2 \leq \dots \leq x_n$, we define

$$\begin{aligned}
 K(k, n) &= \sum_{i=1}^k v(x_i)w^\beta(x_i) \sum_{i=1}^k u(x_i)w^{-\beta}(x_i) \\
 &+ \sum_{i=1}^k v(x_i)w^{-\alpha}(x_i) \sum_{i=1}^k u(x_i)w^\alpha(x_i) + \sum_{i=k+1}^n v(x_i)w^\alpha(x_i) \sum_{i=1}^n u(x_i)w^{-\alpha}(x_i) \\
 &+ \sum_{i=1}^k v(x_i)w^\alpha(x_i) \sum_{i=k+1}^n u(x_i)w^{-\alpha}(x_i) + \sum_{i=k+1}^n v(x_i)w^{-\beta}(x_i) \sum_{i=1}^n u(x_i)w^\beta(x_i) \\
 &+ \sum_{i=1}^k v(x_i)w^{-\beta}(x_i) \sum_{i=k+1}^n u(x_i)w^\beta(x_i),
 \end{aligned}$$

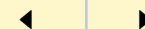
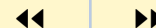
and

$$\begin{aligned}
 L(k, n) &= \sum_{i=1}^k v(x_i)w^\beta(x_i) \sum_{i=1}^k u(x_i)w^\alpha(x_i) \\
 &+ \sum_{i=k+1}^n v(x_i)w^\alpha(x_i) \sum_{i=1}^n u(x_i)w^\beta(x_i) \\
 &+ \sum_{i=1}^k v(x_i)w^\alpha(x_i) \sum_{i=k+1}^n u(x_i)w^\beta(x_i),
 \end{aligned}$$



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where, $k = 1, 2, \dots, n$,

$$\sum_{i=n+1}^n u(x_i)w^\beta(x_i) = \sum_{i=n+1}^n v(x_i)w^\alpha(x_i) = 0,$$

$$\sum_{i=n+1}^n u(x_i)w^{-\alpha}(x_i) = \sum_{i=n+1}^n v(x_i)w^{-\beta}(x_i) = 0.$$

Proposition 4.1. *Let w and u/v be both increasing or both decreasing. If $\alpha > \beta$, then we have*

$$(4.1) \quad \sum_{i=1}^n v(x_i)w^\alpha(x_i) \sum_{i=1}^n u(x_i)w^{-\alpha}(x_i) + \sum_{i=1}^n v(x_i)w^{-\beta}(x_i) \sum_{i=1}^n u(x_i)w^\beta(x_i)$$

$$= K(1, n) \leq \dots \leq K(k, n) \leq K(k+1, n) \leq \dots \leq K(n, n)$$

$$= \sum_{i=1}^n v(x_i)w^\beta(x_i) \sum_{i=1}^n u(x_i)w^{-\beta}(x_i) + \sum_{i=1}^n v(x_i)w^{-\alpha}(x_i) \sum_{i=1}^n u(x_i)w^\alpha(x_i)$$

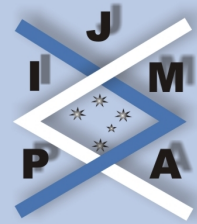
and

$$(4.2) \quad \sum_{i=1}^n v(x_i)w^\alpha(x_i) \sum_{i=1}^n u(x_i)w^\beta(x_i)$$

$$= L(1, n) \leq \dots \leq L(k, n) \leq L(k+1, n) \leq \dots \leq L(n, n)$$

$$= \sum_{i=1}^n v(x_i)w^\beta(x_i) \sum_{i=1}^n u(x_i)w^\alpha(x_i).$$

If $\alpha < \beta$, then the inequalities in (4.1) and (4.2) are reversed.



Proof. Replacing p_i, q_i, a_i and b_i in (2.1) (or the reverse of (2.1)) with $v(x_i)w^\beta(x_i), v(x_i)w^{-\alpha}(x_i), w^{\alpha-\beta}(x_i)$ and $u(x_i)/v(x_i)$, respectively, we obtain (4.1) (or the reverse of (4.1)). Replacing p_i, a_i and b_i in (2.2) (or the reverse of (2.2)) with $v(x_i)w^\beta(x_i), w^{\alpha-\beta}(x_i)$ and $u(x_i)/v(x_i)$, respectively, we obtain (4.2) (or the reverse of (4.2)).

This completes the proof of Proposition 4.1. \square

Remark 1. (4.1) and (4.2) are generated by Proposition 1 in [4].

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous convex function with $f'_+(a)$ ($= f'_-(a)$ is assumed) and $f'_-(b)$, $\{f(x) | x \in [a, b]\} = [d, e]$. Also, let $h : [d, e] \rightarrow (0, +\infty)$ be an integrable function, and $g : [d, e] \rightarrow \mathbb{R}$ be a strict monotonic function. We define

$$(4.3) \quad E(g; f, h) = g^{-1} \left[\left(\int_a^b h(f(t)) f'_-(t) dt \right)^{-1} \int_a^b h(f(t)) g(f(t)) f'_-(t) dt \right],$$

$$(4.4) \quad M(g; f, h) = g^{-1} \left[\left(\int_a^b h(f(t)) dt \right)^{-1} \int_a^b h(f(t)) g(f(t)) dt \right],$$

$$(4.5) \quad R(x; g; f, h) = g^{-1} \left[\left(\int_a^b h(f(t)) dt \int_a^b h(f(t)) f'_-(t) dt \right)^{-1} C_0(x; h(f); g(f), f'_-) \right]$$

and

$$(4.6) \quad \tilde{R}(y; g; f, h) = g^{-1} \left[\left(\int_a^b h(f(t)) dt \int_a^b h(f(t)) f'_-(t) dt \right)^{-1} \tilde{C}_0(y; h(f); g(f), f'_-) \right].$$

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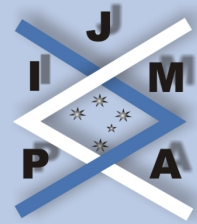
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Proposition 4.2. *If f is monotone, Then we have*

1. $R(x; g; f, h)$ is increasing on $[a, b]$ with x . For $x \in [a, b]$ we have

$$(4.7) \quad \begin{aligned} M(g; f, h) &= R(a; g; f, h) \\ &\leq R(x; g; f, h) \leq R(b; g; f, h) = E(g; f, h). \end{aligned}$$

2. $\tilde{R}(y; g; f, h)$ is decreasing on $[a, b]$ with y . For $y \in [a, b]$ we have

$$(4.8) \quad \begin{aligned} M(g; f, h) &= \tilde{R}(b; g; f, h) \\ &\leq \tilde{R}(y; g; f, h) \leq \tilde{R}(a; g; f, h) = E(g; f, h). \end{aligned}$$

Proof. From the convexity of f , we get that $f'_-(t)$ is increasing on $[a, b]$ and the integrals in $E(g; f, h)$, $R(x; g; f, h)$ and $\tilde{R}(y; g; f, h)$ are valid (see [5]). From $h(x) > 0$, $x \in [d, e]$, we have

$$(4.9) \quad \int_a^b h(f(t)) dt > 0.$$

From the convexity of f , when $f(a) < f(b)$ or $f(a) > f(b)$, Wang in [5] proved that

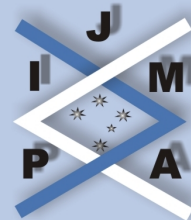
$$(4.10) \quad \int_a^b h(f(t)) f'_-(t) dt > 0$$

or

$$(4.11) \quad \int_a^b h(f(t)) f'_-(t) dt < 0.$$

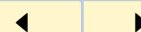
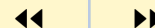
(1) Let us first assume that g is a strictly increasing function.

Case 1. From the increasing properties of f , we have $f(a) < f(b)$. Further, (4.10)



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holds. To prove that $R(x; g; f, h)$ is increasing, from (4.5), (4.9) and (4.10), we only need to prove that

$$(4.12) \quad C_0\left(x; h(f); g(f), f'_-\right) \\ = \left(\int_a^b h(f(t)) dt \int_a^b h(f(t)) f'_-(t) dt \right) g\left(R(x; g; f, h)\right)$$

is increasing on $[a, b]$ with x .

Indeed, since f is increasing on $[a, b]$, we have that $g(f(t))$ is increasing on $[a, b]$.

By Theorem 2.2, $C_0(x; h(f); g(f), f'_-)$ is monotonically increasing with $x \in [a, b]$.

For $x \in [a, b]$, from (4.3), (4.4), (4.5), (4.9) and (4.10), then (4.7) is equivalent to

$$(4.13) \quad \int_a^b h(f(t)) g(f(t)) dt \int_a^b h(f(t)) f'_-(t) dt \\ = C_0(a; h(f); g(f), f'_-) \leq C_0(x; h(f); g(f), f'_-) \leq C_0(b; h(f); g(f), f'_-) \\ = \int_a^b h(f(t)) dt \int_a^b h(f(t)) g(f(t)) f'_-(t) dt.$$

Replacing $p(t)$, $r(t)$ and $s(t)$ in (2.5) with $h(f(t))$, $g(f(t))$ and $f'_-(t)$, respectively, we obtain (4.13).

Case 2. If f is decreasing on $[a, b]$, then we have $f(a) > f(b)$, i.e. (4.11) holds. To prove that $R(x; g; f, h)$ is increasing, from (4.5), (4.9) and (4.11), we only need to prove that $C_0(x; h(f); g(f), f'_-)$ (see (4.12)) is decreasing on $[a, b]$ with x .

Indeed, since f is decreasing on $[a, b]$, then $g(f(t))$ is decreasing on $[a, b]$. By Theorem 2.2, $C_0(x; h(f); g(f), f'_-)$ is decreasing with $x \in [a, b]$.

For $x \in [a, b]$, from (4.3), (4.4), (4.5), (4.9) and (4.11), then (4.7) is equivalent to the reverse of (4.13). Replacing $p(t)$, $r(t)$ and $s(t)$ in the reverse of (2.5) with $h(f(t))$, $g(f(t))$ and $f'_-(t)$, respectively, we obtain the reverse of (4.13).



The second case: g is a strictly decreasing function. Using the same arguments for g as a strictly increasing function, we can also prove (1).

(2) Using the same arguments as those for (1), with (2.6) and (2.7), we can prove that $\tilde{R}(x; g; f, h)$ is decreasing on $[a, b]$ with x , and (4.8) holds.

This completes the proof of Proposition 4.2. □

Remark 2. (4.7)–(4.8) can be generated by (*) in [6] or Proposition 8.1 in [5].

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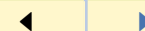
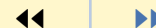


Mappings Related to
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