THREE MAPPINGS RELATED TO CHEBYSHEV-TYPE INEQUALITIES

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Abstract: In this paper, by the Chebyshev-type inequalities we define three mappings, investigate their main properties, give some refinements for Chebyshev-type inequalities, obtain some applications.

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Contents

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1. Introduction

Let $n(\geq 2)$ be a given positive integer, $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ be known as sequences of real numbers. Also, let $p_i > 0$ and $q_i > 0$ $(i = 1, 2, \dots, n)$, $P_j = p_1 + p_2 + \dots + p_j$ and $Q_j = q_1 + q_2 + \dots + q_j$ $(j = 1, 2, \dots, n)$.

If A and B are both increasing or both decreasing, then

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \leq n \sum_{i=1}^{n} a_i b_i. \tag{1.1}$$

If one of the sequences A or B is increasing and the other decreasing, then the inequality (1.1) is reversed.

The inequality (1.1) is called the Chebyshev's inequality, see [1, 2].

For A and B both increasing or both decreasing, Behdzet in [3] extended inequality (1.1) to

$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} q_i b_i\right) + \left(\sum_{i=1}^{n} q_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) \leq P_n \sum_{i=1}^{n} q_i a_i b_i + Q_n \sum_{i=1}^{n} p_i a_i b_i. \tag{1.2}$$

If one of the sequences A or B is increasing and the other decreasing, then the inequality (1.2) is reversed.

For $p_i = q_i$, $i = 1, 2, \dots, n$, the inequality (1.2) reduces to

$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) \leq P_n \sum_{i=1}^{n} p_i a_i b_i, \tag{1.3}$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

where, A and B are both increasing or both decreasing. If one of the sequences A or B is increasing and the other decreasing, then the inequality (1.3) is reversed.

Let $r, s : [a, b] \to \mathbb{R}$ be integrable functions, either both increasing or both decreasing. Furthermore, let $p, q : [a, b] \to [0, +\infty)$ be the integrable functions. Then

$$\begin{aligned}(1.4)\quad \int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt \\ \leq \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt.\end{aligned}$$

If one of the functions r or s is increasing and the other decreasing, then the inequality (1.4) is reversed.

When $p(t) = q(t)$, $t \in [a, b]$, the inequality (1.4) reduces to

$$(1.5)\quad \int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt,$$

where r and s are both increasing or both decreasing. If one of the functions r or s is increasing and the other decreasing, then the inequality (1.5) is reversed.

Inequalities (1.4) and (1.5) are the integral forms of inequalities (1.2) and (1.3), respectively (see [1, 2]).

The results from other inequalities connected with (1.1) to (1.5) can be seen in [1], [3] – [8] and [2, pp. 61–65].

We define three mappings c, C and $\widetilde{C}$ by $c : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R}$,

$$(1.6)\quad c(k, n; p_i, q_i) = P_k \sum_{i=1}^{k} q_i a_i b_i + Q_k \sum_{i=1}^{k} p_i a_i b_i$$

journal of inequalities in pure and applied mathematics
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$$+\left(\sum_{i=k+1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} q_i b_i\right)+\left(\sum_{i=1}^{k} p_i a_i\right)\left(\sum_{i=k+1}^{n} q_i b_i\right)$$
$$+\left(\sum_{i=k+1}^{n} q_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right)+\left(\sum_{i=1}^{k} q_i a_i\right)\left(\sum_{i=k+1}^{n} p_i b_i\right),$$

where $k = 1, 2, \dots, n$, and

$$\sum_{i=n+1}^{n} q_i a_i = \sum_{i=n+1}^{n} p_i b_i = \sum_{i=n+1}^{n} p_i a_i = \sum_{i=n+1}^{n} q_i b_i = 0$$

is assumed.

For $C : [a, b] \to \mathbb{R}$,

$$(1.7)\quad C(x; p, q; r, s) = \int_a^x p(t)dt \int_a^x q(t)r(t)s(t)dt + \int_a^x q(t)dt \int_a^x p(t)r(t)s(t)dt$$
$$+ \int_x^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^x p(t)r(t)dt \int_x^b q(t)s(t)dt$$
$$+ \int_x^b q(t)r(t)dt \int_a^b p(t)s(t)dt + \int_a^x q(t)r(t)dt \int_x^b p(t)s(t)dt$$

and for $\widetilde{C} : [a, b] \to \mathbb{R}$,

$$(1.8)\quad \widetilde{C}(y; p, q; r, s) = \int_y^b p(t)dt \int_y^b q(t)r(t)s(t)dt + \int_y^b q(t)dt \int_y^b p(t)r(t)s(t)dt$$

vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$+\int_a^y p(t)r(t)dt\int_a^b q(t)s(t)dt+\int_y^b p(t)r(t)dt\int_a^y q(t)s(t)dt$$
$$+\int_a^y q(t)r(t)dt\int_a^b p(t)s(t)dt+\int_y^b q(t)r(t)dt\int_a^y p(t)s(t)dt.$$

We write

(1.9)
$$c_1(k,n;p_i) = \frac{1}{2}c(k,n;p_i,p_i) = P_k\sum_{i=1}^k p_ia_ib_i+\left(\sum_{i=k+1}^n p_ia_i\right)\left(\sum_{i=1}^n p_ib_i\right)+\left(\sum_{i=1}^k p_ia_i\right)\left(\sum_{i=k+1}^n p_ib_i\right),$$

(1.10)
$$c_2(k,n)=c_1(k,n;1) = k\sum_{i=1}^k a_ib_i+\left(\sum_{i=k+1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)+\left(\sum_{i=1}^k a_i\right)\left(\sum_{i=k+1}^n b_i\right),$$

(1.11)
$$C_0(x;p;r,s) = \frac{1}{2}C(x;p,p;r,s) = \int_a^x p(t)dt\int_a^x p(t)r(t)s(t)dt+\int_x^b p(t)r(t)dt\int_a^b p(t)s(t)dt + \int_a^x p(t)r(t)dt\int_x^b p(t)s(t)dt$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

and

$$\widetilde{C}_0(y;p;r,s) = \frac{1}{2}\widetilde{C}(y;p,p;r,s) = \int_y^b p(t)dt \int_y^b p(t)r(t)s(t)dt + \int_a^y p(t)r(t)dt \int_a^b p(t)s(t)dt + \int_y^b p(t)r(t)dt \int_a^y p(t)s(t)dt. \tag{1.12}$$

(1.10), (1.6), (1.9), (1.7) and (1.8), (1.11) and (1.12) are generated by the inequalities (1.1) to (1.5), respectively.

The aim of this paper is to study the monotonicity properties of c, C and $\widetilde{C}$, and obtain some refinements of (1.1) to (1.5) using these monotonicity properties. Some applications are given.

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Main Results

The monotonicity properties of the mapping c, c_1 and c_2 are embodied in the following theorem.

Theorem 2.1. *Let c, c_1 and c_2 be defined as in the first section. If A and B are both increasing or both decreasing, then we have the following refinements of (1.2), (1.3) and (1.1)*

$$\left(\sum_{i=1}^n p_i a_i\right)\left(\sum_{i=1}^n q_i b_i\right)+\left(\sum_{i=1}^n q_i a_i\right)\left(\sum_{i=1}^n p_i b_i\right) \tag{2.1}$$
$$= c(1,n;p_i,q_i) \le \cdots \le c(k,n;p_i,q_i) \le c(k+1,n;p_i,q_i) \le \cdots$$
$$\le c(n,n;p_i,q_i) = P_n \sum_{i=1}^n q_i a_i b_i + Q_n \sum_{i=1}^n p_i a_i b_i,$$

$$\left(\sum_{i=1}^n p_i a_i\right)\left(\sum_{i=1}^n p_i b_i\right) = c_1(1,n;p_i) \le \cdots \le c_1(k,n;p_i) \tag{2.2}$$
$$\le c_1(k+1,n;p_i) \le \cdots \le c_1(n,n;p_i)$$
$$= P_n \sum_{i=1}^n p_i a_i b_i$$

and

$$\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right) = c_2(1,n) \le \cdots \le c_2(k,n) \tag{2.3}$$
$$\le c_2(k+1,n) \le \cdots \le c_2(n,n) = n\sum_{i=1}^n a_i b_i,$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

respectively. If one of the sequences A or B is increasing and the other decreasing, then inequalities in (2.1)–(2.3) are reversed.

The monotonicity properties of the mappings C and C_0 are given in the following theorem.

Theorem 2.2. *Let C and C_0 be defined as in the first section. If r and s are both increasing or both decreasing, then $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$ are increasing on $[a,b]$ with x, and for $x \in [a,b]$ we have the following refinements of (1.4) and (1.5)*

$$\begin{aligned}(2.4)\qquad &\int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt \\ &= C(a;p,q;r,s) \le C(x;p,q;r,s) \le C(b;p,q;r,s) \\ &= \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad \int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt &= C_0(a;p;r,s) \\ &\le C_0(x;p;r,s) \le C_0(b;p;r,s) \\ &= \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt,\end{aligned}$$

respectively. If one of the functions r or s is increasing and the other decreasing, then $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$ are decreasing on $[a,b]$ with x, and inequalities in (2.4) and (2.5) are reversed.

The monotonicity properties of $\widetilde{C}$ and $\widetilde{C_0}$ are given in the following theorem.

vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 2.3. *Let $\widetilde{C}$ and $\widetilde{C_0}$ be defined as in the first section. If r and s are both increasing or both decreasing, then $\widetilde{C}(y;p,q;r,s)$ and $\widetilde{C}_0(y;p;r,s)$ are decreasing on $[a,b]$ with y, and for $y \in [a,b]$ we have the following refinements of (1.4) and (1.5)*

$$\int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt$$
$$= \widetilde{C}(b;p,q;r,s) \le \widetilde{C}(y;p,q;r,s) \le \widetilde{C}(a;p,q;r,s)$$
$$= \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt \tag{2.6}$$

and

$$\int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt = \widetilde{C}_0(b;p;r,s) \le \widetilde{C}_0(y;p;r,s) \le \widetilde{C}_0(a;p;r,s)$$
$$= \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt, \tag{2.7}$$

respectively. If one of the functions r or s is increasing and the other decreasing, then $\widetilde{C}(y;p,q;r,s)$ and $\widetilde{C_0}(y;p;r,s)$ are increasing on $[a,b]$ with y, and the inequalities in (2.6) and (2.7) are reversed.

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Proof of Theorems

Proof of Theorem 2.1. For $k = 2, 3, \dots, n$, we have

$$
\begin{aligned}
(3.1)\qquad & c(k,n;p_i,q_i) - c(k-1,n;p_i,q_i) \\
&= (P_{k-1}+p_k)\left(\sum_{i=1}^{k-1} q_i a_i b_i + q_k a_k b_k\right) \\
&\quad + (Q_{k-1}+q_k)\left(\sum_{i=1}^{k-1} p_i a_i b_i + p_k a_k b_k\right) \\
&\quad - \left[P_{k-1}\sum_{i=1}^{k-1} q_i a_i b_i + Q_{k-1}\sum_{i=1}^{k-1} p_i a_i b_i\right] + \sum_{i=k+1}^{n} p_i a_i \sum_{i=1}^{n} q_i b_i \\
&\quad + \left(p_k a_k + \sum_{i=1}^{k-1} p_i a_i\right)\sum_{i=k+1}^{n} q_i b_i - \left(p_k a_k + \sum_{i=k+1}^{n} p_i a_i\right)\sum_{i=1}^{n} q_i b_i \\
&\quad - \sum_{i=1}^{k-1} p_i a_i \left(q_k b_k + \sum_{i=k+1}^{n} q_i b_i\right) + \sum_{i=k+1}^{n} q_i a_i \sum_{i=1}^{n} p_i b_i \\
&\quad + \left(q_k a_k + \sum_{i=1}^{k-1} q_i a_i\right)\sum_{i=k+1}^{n} p_i b_i - \left(q_k a_k + \sum_{i=k+1}^{n} q_i a_i\right)\sum_{i=1}^{n} p_i b_i \\
&\quad - \sum_{i=1}^{k-1} q_i a_i \left(p_k b_k + \sum_{i=k+1}^{n} p_i b_i\right) \\
&= \left[p_k \sum_{i=1}^{k-1} q_i a_i b_i + p_k a_k b_k \sum_{i=1}^{k-1} q_i - p_k a_k \sum_{i=1}^{k-1} q_i b_i - p_k b_k \sum_{i=1}^{k-1} q_i a_i\right]
\end{aligned}
$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$+\left[q_k\sum_{i=1}^{k-1}p_ia_ib_i+q_ka_kb_k\sum_{i=1}^{k-1}p_i-q_ka_k\sum_{i=1}^{k-1}p_ib_i-q_kb_k\sum_{i=1}^{k-1}p_ia_i\right]$$
$$=p_k\sum_{i=1}^{k-1}q_i\left(a_k-a_i\right)\left(b_k-b_i\right)+q_k\sum_{i=1}^{k-1}p_i\left(a_k-a_i\right)\left(b_k-b_i\right).$$

If A and B are both increasing or both decreasing, then

$$\text{(3.2)}\qquad \Big(a_k-a_i\Big)\Big(b_k-b_i\Big)\geq 0,\quad (i=1,2,\dots,k-1).$$

Using (1.6), (3.1) and (3.2), we obtain (2.1).

If one of the sequences A or B is increasing and the other decreasing, then (3.2) is reversed, which implies that the inequalities in (2.1) are reversed.

For $i=1,2,\dots,n$, replacing q_i in (2.1) with p_i and replacing p_i in (2.2) with 1, we obtain (2.2) and (2.3), respectively. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. For any $x_1,x_2\in[a,b]$, $x_1<x_2$, we write

$$I_1=\int_{x_1}^{x_2}p(t)dt\int_{x_1}^{x_2}q(t)r(t)s(t)dt+\int_{x_1}^{x_2}q(t)dt\int_{x_1}^{x_2}p(t)r(t)s(t)dt$$
$$-\int_{x_1}^{x_2}p(t)r(t)dt\int_{x_1}^{x_2}q(t)s(t)dt-\int_{x_1}^{x_2}q(t)r(t)dt\int_{x_1}^{x_2}p(t)s(t)dt.$$

For $t\in[a,x_1]$, $u\in[x_1,x_2]$, using the properties of double integrals, we get

$$I_2=\iint_{[a,x_1]\times[x_1,x_2]}p(t)q(u)\Big(r(t)-r(u)\Big)\Big(s(t)-s(u)\Big)dtdu$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= \int_a^{x_1} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_a^{x_1} p(t)r(t)s(t)dt \int_{x_1}^{x_2} q(t)dt$$
$$- \int_a^{x_1} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_a^{x_1} p(t)s(t)dt \int_{x_1}^{x_2} q(t)r(t)dt$$

and

$$I_3 = \iint_{[a,x_1]\times[x_1,x_2]} p(u)q(t)\Big(r(t)-r(u)\Big)\Big(s(t)-s(u)\Big)dtdu$$
$$= \int_a^{x_1} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt + \int_a^{x_1} q(t)r(t)s(t)dt \int_{x_1}^{x_2} p(t)dt$$
$$- \int_a^{x_1} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt - \int_a^{x_1} q(t)s(t)dt \int_{x_1}^{x_2} p(t)r(t)dt.$$

When $x_1 = a$, from (1.7), we get

$$C(x_2; p, q; r, s) - C(x_1; p, q; r, s) \tag{3.3}$$
$$= \int_{x_1}^{x_2} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_{x_1}^{x_2} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt$$
$$- \int_{x_1}^{x_2} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_{x_1}^{x_2} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt$$
$$= I_1.$$

When $x_1 > a$, from (1.7), we have

$$C(x_2; p, q; r, s) - C(x_1; p, q; r, s) \tag{3.4}$$

journal of inequalities
in pure and applied
mathematics
issn: 1443-5756

$$
\begin{aligned}
&= \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) p(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) q(t)r(t)s(t)dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) q(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) p(t)r(t)s(t)dt \\
&\quad - \int_a^{x_1} p(t)dt \int_a^{x_1} q(t)r(t)s(t)dt - \int_a^{x_1} q(t)dt \int_a^{x_1} p(t)r(t)s(t)dt \\
&\quad + \int_{x_2}^{b} p(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)s(t)dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) p(t)r(t)dt \int_{x_2}^{b} q(t)s(t)dt \\
&\quad - \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)s(t)dt \\
&\quad - \int_a^{x_1} p(t)r(t)dt \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)s(t)dt \\
&\quad + \int_{x_2}^{b} q(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)s(t)dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) q(t)r(t)dt \int_{x_2}^{b} p(t)s(t)dt \\
&\quad - \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)s(t)dt \\
&\quad - \int_a^{x_1} q(t)r(t)dt \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)s(t)dt
\end{aligned}
$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$
\begin{aligned}
&= \Big[\int_{x_1}^{x_2} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_{x_1}^{x_2} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt \\
&\quad - \int_{x_1}^{x_2} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_{x_1}^{x_2} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt\Big] \\
&\quad + \Big[\int_{a}^{x_1} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_{a}^{x_1} p(t)r(t)s(t)dt \int_{x_1}^{x_2} q(t)dt \\
&\quad - \int_{a}^{x_1} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_{a}^{x_1} p(t)s(t)dt \int_{x_1}^{x_2} q(t)r(t)dt\Big] \\
&\quad + \Big[\int_{a}^{x_1} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt + \int_{a}^{x_1} q(t)r(t)s(t)dt \int_{x_1}^{x_2} p(t)dt \\
&\quad - \int_{a}^{x_1} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt - \int_{a}^{x_1} q(t)s(t)dt \int_{x_1}^{x_2} p(t)r(t)dt\Big] \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

(1) If r and s are both increasing or both decreasing, then we have

$$
\Big(r(t) - r(u)\Big)\Big(s(t) - s(u)\Big) \geq 0, \tag{3.5}
$$

i.e., $I_2 \geq 0$ and $I_3 \geq 0$. By the inequality (1.4), $I_1 \geq 0$ holds . Using (3.3) and (3.4), we obtain that $C(x; p, q; r, s)$ is increasing on $[a, b]$ with x. Further, from (1.11), we get that $C_0(x; p; r, s)$ is increasing on $[a, b]$ with x.

From (1.7) and (1.11), using the increasing properties of $C(x; p, q; r, s)$ and $C_0(x; p; r, s)$, we obtain (2.4) and (2.5), respectively.

(2) If one of the functions r or s is increasing and the other decreasing, then the inequality in (3.5) is reversed, which implies that $I_2 \leq 0$ and $I_3 \leq 0$. By the reverse of (1.4), $I_1 \leq 0$ holds. From (3.3) and (3.4), (1.11), we obtain that $C(x; p, q; r, s)$, $C_0(x; p; r, s)$ are decreasing on $[a, b]$ with x, respectively.

vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

From (1.7) and (1.11), using the decreasing properties of $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$, we obtain the reverse of (2.4) and (2.5), respectively.

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Using the same arguments as those in the proof of Theorem 2.2, we can prove Theorem 2.3. □

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

4. Applications

Let $\mathbf{I}$ be a real interval and $u, v, w : \mathbf{I} \to [0, +\infty)$. For any $\alpha, \beta \in \mathbb{R}$ and any $x_i \in \mathbf{I}$ $(i = 1, 2, \dots, n, n \geq 2)$ satisfying $x_1 \leq x_2 \leq \cdots \leq x_n$, we define

$$\begin{aligned} K(k,n) = &\sum_{i=1}^{k} v(x_i)w^{\beta}(x_i) \sum_{i=1}^{k} u(x_i)w^{-\beta}(x_i) \\ &+ \sum_{i=1}^{k} v(x_i)w^{-\alpha}(x_i) \sum_{i=1}^{k} u(x_i)w^{\alpha}(x_i) + \sum_{i=k+1}^{n} v(x_i)w^{\alpha}(x_i) \sum_{i=1}^{n} u(x_i)w^{-\alpha}(x_i) \\ &+ \sum_{i=1}^{k} v(x_i)w^{\alpha}(x_i) \sum_{i=k+1}^{n} u(x_i)w^{-\alpha}(x_i) + \sum_{i=k+1}^{n} v(x_i)w^{-\beta}(x_i) \sum_{i=1}^{n} u(x_i)w^{\beta}(x_i) \\ &+ \sum_{i=1}^{k} v(x_i)w^{-\beta}(x_i) \sum_{i=k+1}^{n} u(x_i)w^{\beta}(x_i), \end{aligned}$$

and

$$\begin{aligned} L(k,n) = &\sum_{i=1}^{k} v(x_i)w^{\beta}(x_i) \sum_{i=1}^{k} u(x_i)w^{\alpha}(x_i) \\ &+ \sum_{i=k+1}^{n} v(x_i)w^{\alpha}(x_i) \sum_{i=1}^{n} u(x_i)w^{\beta}(x_i) \\ &+ \sum_{i=1}^{k} v(x_i)w^{\alpha}(x_i) \sum_{i=k+1}^{n} u(x_i)w^{\beta}(x_i), \end{aligned}$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

where, $k = 1, 2, \ldots, n$,

$$\sum_{i=n+1}^{n} u(x_i)w^{\beta}(x_i) = \sum_{i=n+1}^{n} v(x_i)w^{\alpha}(x_i) = 0,$$
$$\sum_{i=n+1}^{n} u(x_i)w^{-\alpha}(x_i) = \sum_{i=n+1}^{n} v(x_i)w^{-\beta}(x_i) = 0.$$

Proposition 4.1. *Let w and u/v be both increasing or both decreasing. If $\alpha > \beta$, then we have*

$$\begin{aligned}(4.1)\quad &\sum_{i=1}^{n} v(x_i)w^{\alpha}(x_i)\sum_{i=1}^{n} u(x_i)w^{-\alpha}(x_i) + \sum_{i=1}^{n} v(x_i)w^{-\beta}(x_i)\sum_{i=1}^{n} u(x_i)w^{\beta}(x_i)\\ &= K(1,n) \le \cdots \le K(k,n) \le K(k+1,n) \le \cdots \le K(n,n)\\ &= \sum_{i=1}^{n} v(x_i)w^{\beta}(x_i)\sum_{i=1}^{n} u(x_i)w^{-\beta}(x_i) + \sum_{i=1}^{n} v(x_i)w^{-\alpha}(x_i)\sum_{i=1}^{n} u(x_i)w^{\alpha}(x_i)\end{aligned}$$

and

$$\begin{aligned}(4.2)\quad &\sum_{i=1}^{n} v(x_i)w^{\alpha}(x_i)\sum_{i=1}^{n} u(x_i)w^{\beta}(x_i)\\ &= L(1,n) \le \cdots \le L(k,n) \le L(k+1,n) \le \cdots \le L(n,n)\\ &= \sum_{i=1}^{n} v(x_i)w^{\beta}(x_i)\sum_{i=1}^{n} u(x_i)w^{\alpha}(x_i).\end{aligned}$$

If $\alpha < \beta$, then the inequalities in (4.1) and (4.2) are reversed.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. Replacing p_i, q_i, a_i and b_i in (2.1) (or the reverse of (2.1)) with $v(x_i)w^{\beta}(x_i)$, $v(x_i)w^{-\alpha}(x_i)$, $w^{\alpha-\beta}(x_i)$ and $u(x_i)/v(x_i)$, respectively, we obtain (4.1) (or the reverse of (4.1)). Replacing p_i, a_i and b_i in (2.2) (or the reverse of (2.2)) with $v(x_i)w^{\beta}(x_i)$, $w^{\alpha-\beta}(x_i)$ and $u(x_i)/v(x_i)$, respectively, we obtain (4.2) (or the reverse of (4.2)).

This completes the proof of Proposition 4.1. □

Remark 1. (4.1) and (4.2) are generated by Proposition 1 in [4].

Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function with $f'_+(a)$ ($= f'_-(a)$ is assumed) and $f'_-(b)$, $\{f(x)|x \in [a,b]\} = [d,e]$. Also, let $h : [d,e] \to (0,+\infty)$ be an integrable function, and $g : [d,e] \to \mathbb{R}$ be a strict monotonic function. We define

$$E(g;f,h) = g^{-1}\left[\left(\int_a^b h(f(t))f'_-(t)dt\right)^{-1}\int_a^b h(f(t))g(f(t))f'_-(t)dt\right], \tag{4.3}$$

$$M(g;f,h) = g^{-1}\left[\left(\int_a^b h(f(t))dt\right)^{-1}\int_a^b h(f(t))g(f(t))dt\right], \tag{4.4}$$

$$\begin{aligned} &R(x;g;f,h) \\ &= g^{-1}\left[\left(\int_a^b h(f(t))dt\int_a^b h(f(t))f'_-(t)dt\right)^{-1} C_0\Big(x;h(f);g(f),f'_-\Big)\right] \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\widetilde{R}(y;g;f,h) \\ &= g^{-1}\left[\left(\int_a^b h(f(t))dt\int_a^b h(f(t))f'_-(t)dt\right)^{-1} \widetilde{C}_0\Big(y;h(f);g(f),f'_-\Big)\right]. \end{aligned} \tag{4.6}$$

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proposition 4.2. *If f is monotone, Then we have*

1. *$R(x; g; f, h)$ is increasing on $[a, b]$ with x. For $x \in [a, b]$ we have*

$$M(g; f, h) = R(a; g; f, h) \leq R(x; g; f, h) \leq R(b; g; f, h) = E(g; f, h). \tag{4.7}$$

2. *$\tilde{R}(y; g; f, h)$ is decreasing on $[a, b]$ with y. For $y \in [a, b]$ we have*

$$M(g; f, h) = \tilde{R}(b; g; f, h) \leq \tilde{R}(y; g; f, h) \leq \tilde{R}(a; g; f, h) = E(g; f, h). \tag{4.8}$$

Proof. From the convexity of f, we get that $f'_-(t)$ is increasing on $[a, b]$ and the integrals in $E(g; f, h)$, $R(x; g; f, h)$ and $\widetilde{R}(y; g; f, h)$ are valid (see [5]). From $h(x) > 0$, $x \in [d, e]$, we have

$$\int_a^b h(f(t))dt > 0. \tag{4.9}$$

From the convexity of f, when $f(a) < f(b)$ or $f(a) > f(b)$, Wang in [5] proved that

$$\int_a^b h(f(t))f'_-(t)dt > 0 \tag{4.10}$$

or

$$\int_a^b h(f(t))f'_-(t)dt < 0. \tag{4.11}$$

(1) Let us first assume that g is a strictly increasing function.

Case 1. From the increasing properties of f, we have $f(a) < f(b)$. Further, (4.10)

Mappings Related to Chebyshev-type Inequalities
Liang-Cheng Wang, Li-Hong Liu and Xiu-Fen Ma
vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

holds. To prove that $R(x; g; f, h)$ is increasing, from (4.5), (4.9) and (4.10), we only need to prove that

$$C_0\Big(x; h(f); g(f), f'_-\Big) = \left(\int_a^b h(f(t))dt \int_a^b h(f(t))f'_-(t)dt\right) g\Big(R(x; g; f, h)\Big) \tag{4.12}$$

is increasing on $[a, b]$ with x.

Indeed, since f is increasing on $[a, b]$, we have that $g(f(t))$ is increasing on $[a, b]$. By Theorem 2.2, $C_0(x; h(f); g(f), f'_-)$ is monotonically increasing with $x \in [a, b]$. For $x \in [a, b]$, from (4.3), (4.4), (4.5), (4.9) and (4.10), then (4.7) is equivalent to

$$\begin{aligned} &\int_a^b h(f(t))g(f(t))dt \int_a^b h(f(t))f'_-(t)dt \\ &= C_0(a; h(f); g(f), f'_-) \le C_0(x; h(f); g(f), f'_-) \le C_0(b; h(f); g(f), f'_-) \\ &= \int_a^b h(f(t))dt \int_a^b h(f(t))g(f(t))f'_-(t)dt. \end{aligned} \tag{4.13}$$

Replacing $p(t)$, $r(t)$ and $s(t)$ in (2.5) with $h(f(t))$, $g(f(t))$ and $f'_-(t)$, respectively, we obtain (4.13).

Case 2. If f is decreasing on $[a, b]$, then we have $f(a) > f(b)$, i.e. (4.11) holds. To prove that $R(x; g; f, h)$ is increasing, from (4.5), (4.9) and (4.11), we only need to prove that $C_0(x; h(f); g(f), f'_-)$ (see (4.12)) is decreasing on $[a, b]$ with x.

Indeed, since f is decreasing on $[a, b]$, then $g(f(t))$ is decreasing on $[a, b]$. By Theorem 2.2, $C_0(x; h(f); g(f), f'_-)$ is decreasing with $x \in [a, b]$.

For $x \in [a, b]$, from (4.3), (4.4), (4.5), (4.9) and (4.11), then (4.7) is equivalent to the reverse of (4.13). Replacing $p(t)$, $r(t)$ and $s(t)$ in the reverse of (2.5) with $h(f(t))$, $g(f(t))$ and $f'_-(t)$, respectively, we obtain the reverse of (4.13).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

The second case: g is a strictly decreasing function. Using the same arguments for g as a strictly increasing function, we can also prove (1).

(2) Using the same arguments as those for (1), with (2.6) and (2.7), we can prove that $\tilde{R}(x; g; f, h)$ is decreasing on $[a, b]$ with x, and (4.8) holds.

This completes the proof of Proposition 4.2. □

Remark 2. (4.7)–(4.8) can be generated by $(*)$ in [6] or Proposition 8.1 in [5].

vol. 9, iss. 4, art. 113, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag. Berlin, 1970.

[2] J.-C. KUANG, *Applied Inequalities*, Shandong Science and Technology Press, 2004. (Chinese).

[3] M. BEHDZET, Some results connected with Cebysev's inequality, *Rad. Mat.*, **1**(2) (1985), 185–190.

[4] L.-C. WANG, On the monotonicity of difference generated by the inequalities of Chebyshev type, *J. Sichuan Univ.*, **39**(3) (2002), 398–403. (Chinese).

[5] L.-C. WANG, *Convex Functions and Their Inequalities*, Sichuan University Press, Chengdu, China, 2001. (Chinese).

[6] B.-N. GUO AND FENG QI, Inequalities for generalized weighted mean values of convex function, *Math. Inequalities Appl.*, **4**(2) (2001), 195–202.

[7] F. QI, L.-H. CUI AND S.-L. XU, Some inequalities constructed by Tchebysheff's integral inequality, *Math. Inequalities Appl.*, **2**(4) (1999), 517–528.

[8] F. QI, S.-L. XU AND L. DEBNATH, A new proof of monotonicity for extended mean values, *J. Math. Math. Sci.*, **22**(2) (1999), 417–421.

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