ON CERTAIN INEQUALITIES FOR MEANS IN TWO VARIABLES

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Abstract: We show that certain inequalities for the means A, G, L, I proved by H.-J. Seiffert [12] as well as by H. Alzer and S.-L. Qiu [3] are consequences of some results of the author [5], [7], [9].

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1. Introduction

The logarithmic and identric means of two positive real numbers a and b with $a \neq b$ are defined by

$$L = L(a,b) = \frac{b-a}{\log b - \log a} \quad \text{and} \quad I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}},$$

respectively. These means have been the subject of much intensive research, partly because they are related to many other important means and partly because these means have applications in physics, economics, meteorology, statistics, etc. For a survey of results, with an extended literature, see [3], [6]. For identities involving these, and other means, see e.g. [8], [10]. In particular, the identity

$$\frac{I(a^2,b^2)}{I(a,b)} = (a^a \cdot b^b)^{\frac{1}{a+b}} = S = S(a,b)$$

leads to the weighted geometric mean of a and b, denoted by $S(a,b)$ in [6], [8], [9].

In paper [12], the following two inequalities are proved

$$G(a,b)\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < I(a,b) < A(a,b)\exp\left(-\frac{1}{6}\left(\frac{b-a}{b+a}\right)^2\right), \tag{1.1}$$

where $a \neq b$, $a,b > 0$.

In [3], the following relation appears among many other ones:

$$G(a,b)\exp\left(\frac{1}{6}\left(\frac{b-a}{b+a}\right)^2\right) < L(a,b) < A(a,b)\exp\left(-\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right), \tag{1.2}$$

$a \neq b$, $a,b > 0$.

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We note that, the right hand side inequality of (1.1) was first proved by the author in 1989 [5]. In that paper the following inequality was also shown:

$$\frac{A^2(a,b)}{I(a^2,b^2)} < \exp\left(-\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right). \tag{1.3}$$

The aim of this note is to prove that the above inequalities are connected to each other by a chain of relations, and that, in fact, all are consequences of (1.3).

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2. Main Results

To begin with, we write all the inequalities in another form. The left and right sides of (1.1) may be written respectively as

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{I(a,b)}{G(a,b)}, \tag{2.1}$$

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{A^2(a,b)}{I^2(a,b)} \tag{2.2}$$

and the inequalities of (1.2) as

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{L^2(a,b)}{G^2(a,b)}, \tag{2.3}$$

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{A(a,b)}{L(a,b)}. \tag{2.4}$$

Finally note that, (1.3) may be written as

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{I(a^2,b^2)}{A^2(a,b)} = \frac{I(a,b)S(a,b)}{A^2(a,b)}. \tag{2.5}$$

Theorem 2.1. *The following chain of implications holds true:*

$$(2.5) \Rightarrow (2.2) \Rightarrow (2.4) \Rightarrow (2.1) \Rightarrow (2.3).$$

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Proof. (2.5)⇒(2.2) means that $\frac{I\cdot S}{A^2} < \frac{A^2}{I^2}$, or $S < \frac{A^4}{I^3}$. This inequality is proved in [9, Theorem 1].

(2.2)⇒(2.4) by $\frac{A^2}{I^2} < \frac{A}{L}$, i.e. $I^2 > A \cdot L$. For this inequality, see [7, Relation (9)].

(2.4)⇒(2.1) by $\frac{A}{L} < \frac{I}{G}$, i.e. $A \cdot G < L \cdot I$, see [1].

(2.1)⇒(2.3) by $\frac{I}{G} < \frac{L^2}{G^2}$, i.e. $\sqrt{GI} < L$, see [2].

Therefore all implications are valid. □

We note that inequality (2.5) was a consequence of an integral inequality due to the author [4], (discovered in 1982), to the effect that:

Theorem 2.2. *Let* $f : [a, b] \to \mathbb{R}$ *be a* $2k$*-times* $(k \geq 1)$ *differentiable function such that* $f^{(2k)}(x) > 0$. *Then*

$$\int_a^b f(x)dx > \sum_{j=0}^{k-1} \frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!} f^{(2j)}\left(\frac{a+b}{2}\right). \tag{2.6}$$

For $k = 2$ we obtain that if f is 4-times differentiable, then

$$\frac{1}{b-a}\int_a^b f(x)dx > f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right). \tag{2.7}$$

Clearly, (2.6) and (2.7) are extensions of the classical Hadamard inequality, which says that, if f is convex on $[a, b]$ then

$$\frac{1}{b-a}\int_a^b f(x)dx > f\left(\frac{a+b}{2}\right). \tag{2.8}$$

Applying (2.7) for $f(x) = x \log x$, and using the identity

$$\int_a^b x \log x dx = \frac{1}{4}(b^2 - a^2) \log I(a^2, b^2) \tag{2.9}$$

(see [6]), we get (2.5). Applying (2.7) to $f(x) = -\log x$, we get (2.2), i.e. the right side of (1.1) (see [5]). For another proof, see [11].

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