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AN INTEGRAL INEQUALITY SIMILAR TO QI'S INEQUALITY

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ABSTRACT. In this note, as a complement of an open problem by F. Qi in the paper [*Several integral inequalities*, J. Inequal. Pure Appl. Math. **1** (2002), no. 2, Art. 54. http://jipam.vu.edu.au/article.php?sid=113. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 9, 1039–1042. http://rgmia.vu.edu.au/v2n7.html], a similar problem is posed and an affirmative answer to it is established.

Key words and phrases: Hölder's inequality, Qi's inequality, Integral inequality.

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The following problem was posed by F. Qi in his paper [6]:

Problem 1. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1} \tag{1}$$

hold for $t > 1$?

This problem has attracted much attention from some mathematicians [5]. Its meanings of probability and statistics is found in [2]. See also [1, 3, 4] and the references therein.

Similar to Problem 1, we propose the following

Problem 2. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x)dx\right)^{1-t} \tag{2}$$

hold for $t < 1$?

Before giving an affirmative answer to Problem 2, we establish the following

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Proposition 1. *Let f and g be nonnegative functions with $0 < m \le f(x)/g(x) \le M < \infty$ on $[a,b]$. Then for $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx, \tag{3}$$

and then

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b f(x) dx \right)^{\frac{1}{q}} \left(\int_a^b g(x) dx \right)^{\frac{1}{p}}. \tag{4}$$

Proof. From Hölder's inequality, we obtain

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \le \left(\int_a^b f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b g(x) dx \right)^{\frac{1}{q}}, \tag{5}$$

that is,

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \le \left(\int_a^b [f(x)]^{\frac{1}{p}} [f(x)]^{\frac{1}{q}} dx \right)^{\frac{1}{p}} \left(\int_a^b [g(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \right)^{\frac{1}{q}}. \tag{6}$$

Since $[f(x)]^{\frac{1}{p}} \le M^{\frac{1}{p}} [g(x)]^{\frac{1}{p}}$ and $[g(x)]^{\frac{1}{q}} \le m^{-\frac{1}{q}} [f(x)]^{\frac{1}{q}}$, from the above inequality it follows that

$$\begin{aligned} &\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \\ &\qquad \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx \right)^{\frac{1}{p}} \left(\int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx \right)^{\frac{1}{q}}, \end{aligned} \tag{7}$$

that is

$$\int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \le M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx. \tag{8}$$

Hence, the inequality (3) is proved.

The inequality (4) follows from substituting the following

$$\int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx \le \left(\int_a^b f(x) dx \right)^{\frac{1}{q}} \left(\int_a^b g(x) dx \right)^{\frac{1}{p}} \tag{9}$$

into (8), which can be obtained by Hölder's inequality. □

Now we are in a position to give an affirmative answer to Problem 2 as follows.

Proposition 2. *For a given positive integer $p \ge 2$, if $0 < m \le f(x) \le M$ on $[a,b]$ with $M \le m^{(p-1)^2} \Big/ (b-a)^p$, then*

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \le \left(\int_a^b f(x) dx \right)^{1-\frac{1}{p}}. \tag{10}$$

Proof. Putting $g(x) \equiv 1$ into (4) yields

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \le K \left(\int_a^b f(x) dx \right)^{1-\frac{1}{p}}, \tag{11}$$

where $K = M^{\frac{1}{p^2}} (b-a)^{\frac{1}{p}} \Big/ m^{\left(1-\frac{1}{p}\right)^2}$.

From $M \leq m^{(p-1)^2} \Big/ (b-a)^p$, we conclude that $K \leq 1$. Thus the inequality (10) is proved. □

Remark 3. Now we discuss a simple case of "equality" in Proposition 2. If we make the substitution $f(x) = M = m$ and $b - a = 1$ with $p = 2$, then the equality in (10) holds.

In order to illustrate a possible practical use of Proposition 2, we shall give in the following two simple examples in which we can apply inequality (10).

Example 1. Let $f(x) = 8x^2$ on $[1/2, 1]$ with $M = 8$ and $m = 2$. Taking $p = 2$, we see that the conditions of Proposition 2 are fulfilled and straightforward computation yields

$$\int_{1/2}^{1} \left(8x^2\right)^{1/2} dx = \frac{3}{4}\sqrt{2} < \left(\int_{1/2}^{1} 8x^2 dx\right)^{\frac{1}{2}} = \frac{\sqrt{7}}{\sqrt{3}} .$$

Example 2. Let $f(x) = e^x$on $[1, 2]$ with $M = e^2$ and $m = e$.
Taking $p = 3$, all the conditions of Proposition 2 are satisfied and direct calculation produces

$$\int_{1}^{2} \left(e^x\right)^{1/3} dx = 3\left(e^{2/3} - e^{1/3}\right) \approx 1.65 < \left(\int_{1}^{2} e^x dx\right)^{\frac{2}{3}} = \left(e^2 - e\right)^{2/3} \approx 2.78.$$

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