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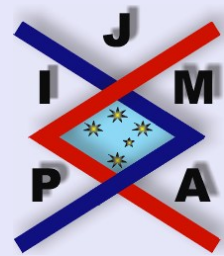
AN INTEGRAL INEQUALITY SIMILAR TO QI'S INEQUALITY

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Abstract

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Abstract

In this note, as a complement of an open problem by F. Qi in the paper [*Several integral inequalities,* J. Inequal. Pure Appl. Math. **1** (2002), no. 2, Art. 54. http://jipam.vu.edu.au/article.php?sid=113. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 9, 1039–1042. http://rgmia.vu.edu.au/v2n7.html], a similar problem is posed and an affirmative answer to it is established.

The following problem was posed by F. Qi in his paper [6]:

Problem 1. *Under what conditions does the inequality*

$$(1) \qquad \int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1}$$

hold for $t > 1$?

This problem has attracted much attention from some mathematicians [5]. Its meanings of probability and statistics is found in [2]. See also [1, 3, 4] and the references therein.

Similar to Problem 1, we propose the following

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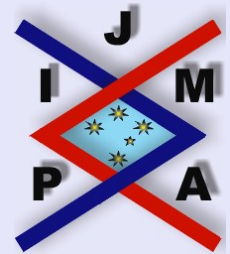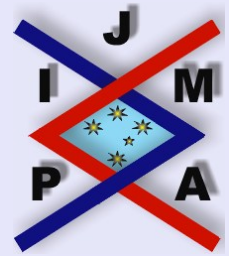
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Problem 2. *Under what conditions does the inequality*

$$(2) \qquad \int_a^b [f(x)]^t dx \leq \left(\int_a^b f(x) dx \right)^{1-t}$$

hold for $t < 1$?

Before giving an affirmative answer to Problem 2, we establish the following

Proposition 1. *Let f and g be nonnegative functions with $0 < m \leq f(x)/g(x) \leq M < \infty$ on $[a, b]$. Then for $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$(3) \qquad \int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}} [g(x)]^{\frac{1}{p}} dx,$$

and then

$$(4) \qquad \int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b f(x) dx \right)^{\frac{1}{q}} \left(\int_a^b g(x) dx \right)^{\frac{1}{p}}.$$

Proof. From Hölder's inequality, we obtain

$$(5) \qquad \int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \leq \left(\int_a^b f(x) dx \right)^{\frac{1}{p}} \left(\int_a^b g(x) dx \right)^{\frac{1}{q}},$$

that is,

$$(6) \qquad \int_a^b [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx$$

$$\leq \left(\int_a^b [f(x)]^{\frac{1}{p}} [f(x)]^{\frac{1}{q}} dx \right)^{\frac{1}{p}} \left(\int_a^b [g(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} dx \right)^{\frac{1}{q}}.$$

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Since $[f(x)]^{\frac{1}{p}} \leq M^{\frac{1}{p}}[g(x)]^{\frac{1}{p}}$ and $[g(x)]^{\frac{1}{q}} \leq m^{-\frac{1}{q}}[f(x)]^{\frac{1}{q}}$, from the above inequality it follows that

$$
\begin{aligned}
(7) \quad &\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\,dx \\
&\leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left(\int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\,dx \right)^{\frac{1}{p}} \left(\int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\,dx \right)^{\frac{1}{q}},
\end{aligned}
$$

that is

$$
(8) \qquad \int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\,dx \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\,dx.
$$

Hence, the inequality (3) is proved.

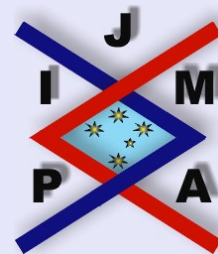The inequality (4) follows from substituting the following

$$
(9) \qquad \int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\,dx \leq \left(\int_a^b f(x)dx \right)^{\frac{1}{q}} \left(\int_a^b g(x)dx \right)^{\frac{1}{p}}
$$

into (8), which can be obtained by Hölder's inequality. $\qquad \square$

Now we are in a position to give an affirmative answer to Problem 2 as follows.

Proposition 2. *For a given positive integer $p \geq 2$, if $0 < m \leq f(x) \leq M$ on $[a, b]$ with $M \leq m^{(p-1)^2} \big/ (b-a)^p$, then*

$$
(10) \qquad \int_a^b [f(x)]^{\frac{1}{p}}\,dx \leq \left(\int_a^b f(x)dx \right)^{1-\frac{1}{p}}.
$$

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Proof. Putting $g(x) \equiv 1$ into (4) yields

$$(11) \qquad \int_a^b [f(x)]^{\frac{1}{p}} dx \le K \left(\int_a^b f(x) dx \right)^{1-\frac{1}{p}},$$

where $K = M^{\frac{1}{p^2}} (b-a)^{\frac{1}{p}} \Big/ m^{\left(1-\frac{1}{p}\right)^2}$.

From $M \le m^{(p-1)^2} \Big/ (b-a)^p$, we conclude that $K \le 1$. Thus the inequality (10) is proved. $\qquad \square$

Remark 1. *Now we discuss a simple case of "equality" in Proposition 2. If we make the substitution $f(x) = M = m$ and $b - a = 1$ with $p = 2$, then the equality in (10) holds.*

In order to illustrate a possible practical use of Proposition 2, we shall give in the following two simple examples in which we can apply inequality (10).

Example 1. *Let $f(x) = 8x^2$ on $[1/2, 1]$ with $M = 8$ and $m = 2$. Taking $p = 2$, we see that the conditions of Proposition 2 are fulfilled and straightforward computation yields*

$$\int_{1/2}^1 \left(8x^2\right)^{1/2} dx = \frac{3}{4}\sqrt{2} < \left(\int_{1/2}^1 8x^2 dx \right)^{\frac{1}{2}} = \frac{\sqrt{7}}{\sqrt{3}} \ .$$

Example 2. *Let $f(x) = e^x$ on $[1, 2]$ with $M = e^2$ and $m = e$.*
Taking $p = 3$, all the conditions of Proposition 2 are satisfied and direct calculation produces

$$\int_1^2 \left(e^x\right)^{1/3} dx = 3 \left(e^{2/3} - e^{1/3}\right) \approx 1.65 < \left(\int_1^2 e^x dx \right)^{\frac{2}{3}} = \left(e^2 - e\right)^{2/3} \approx 2.78.$$
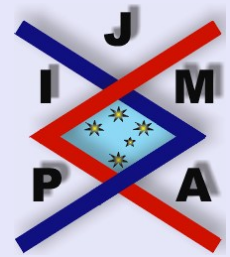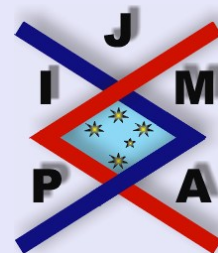
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