Journal of Inequalities in Pure and Applied Mathematics

AN IMPROVEMENT OF THE GRÜSS INEQUALITY

A.McD. MERCER

Department of Mathematics and Statistics
University of Guelph
Guelph, Ontario K8N 2W1, Canada.

EMail: amercer@reach.net

ISSN (electronic): 1443-5756
161-05

volume 6, issue 4, article 93, 2005.

Received 21 May, 2005; accepted 27 July, 2005.

Communicated by: P.S. Bullen

Abstract

The Grüss inequality is improved by adding a positive component to the left hand side.

2000 Mathematics Subject Classification: 26D15.
Key words: Inequality, Stieltjes integral, Linear functional.

This note is dedicated to the memory of my wife Mari.

Contents

1. Introduction

Let $L^\infty(a,b)$ denote the usual Banach algebra of essentially bounded functions defined a.e. on (a,b) and let the functions f and g be members of this set with $m \le f(x) \le M$, $p \le g(x) \le P$ a.e.

Then the classical Grüss inequality [1] reads as follows:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \le \frac{1}{4}(M-m)(P-p)\,. \tag{1.1}$$

Proofs of this inequality and other forms of it can be found, for example, in [2] and Chapter 10 of [3] serves as a comprehensive reference. There are also many references to be found at the web site http://jipam.vu.edu.au.

Since (1.1) is invariant under affine transformations of f and g (i.e. $f \to Af+B, g \to Cg+D$) we could, without any loss of generality, put $M = P = 1$ and $m = p = 0$. It may be noted also that if in (1.1) we were to replace $f(x)$ by $M + m - f(x)$ we would obtain:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx \ge -\frac{1}{4}(M-m)(P-p)$$

so it is immaterial whether we enclose the left hand side of (1.1) within modulus signs or not.

We mention this because the Grüss inequality appears in both forms. Finally, there is no loss in taking the basic interval (a, b) to be $(0, 1)$.

To sum up, the inequality stated as:

$$\int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx \leq \frac{1}{4} \tag{1.2}$$

with $0 \leq f(x), g(x) \leq 1$ a.e. is entirely equivalent to (1.1).

It is our purpose in this note to show that the Grüss inequality (1.2) can be sharpened to the following:

Theorem 1.1. *We have*

$$\begin{aligned} \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx \\ + \int_F |f(x) - g(x)|\, dx \int_G |f(x) - g(x)|\, dx \leq \frac{1}{4}, \end{aligned} \tag{1.3}$$

where F and G are the sets $F = \{x : f(x) \geq g(x)\}$, $G = \{x : f(x) < g(x)\}$.

It is interesting to note that both (1.2) and (1.3) are best-possible in the sense that there are functions which give equality in each. These functions are:

$$f(x) = g(x) = 1 \quad \text{if} \quad 0 < x < \frac{1}{2},$$
$$f(x) = g(x) = 0 \quad \text{if} \quad \frac{1}{2} \leq x < 1.$$

On the other hand, the Grüss inequality (1.2) is not best-possible in a more conventional sense. For, if we take

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le \frac{1}{2} \\ 0 & \quad\text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \frac{1}{2} \quad \text{if} \quad 0 \le x \le 1$$

we find that the left hand side of the Grüss inequality (1.2) is zero whereas the value of the left side of our new inequality (1.3) is $\frac{1}{16}$.

We shall write

$$\int f \quad \text{to mean} \quad \int_0^1 f(x)dx \quad \text{etc.} \tag{1.4}$$

2. Proof of the Theorem

We now prove the result stated in (1.3).

Proof. Let $u(x) = \max[f(x), g(x)]$ and $w(x) = \min[f(x), g(x)]$.

Then $f(x)g(x) = u(x)w(x)$, so that

$$\int fg = \int uw. \tag{2.1}$$

Next

$$\int f \int g - \int u \int w = \left[\int_F f + \int_G f\right]\left[\int_F g + \int_G g\right] - \left[\int_F f + \int_G g\right]\left[\int_F g + \int_G f\right].$$

This reduces to

$$\left[\int_F f - \int_F g\right]\left[\int_G g - \int_G f\right],$$

which equals

$$\int_F |f-g| \int_G |f-g|$$

and so we have

$$\int f \int g - \int u \int w = \int_F |f-g| \int_G |f-g|. \tag{2.2}$$

From (2.1) and (2.2) we get

$$\int fg - \int f \int g + \int_F |f-g| \int_G |f-g| = \int uw - \int u \int w. \tag{2.3}$$

Since $0 < f, g < 1$ then $0 < w \le u < 1$ and so the right hand side here is majorised by

$$\int w - \left(\int w\right)^2 \le \frac{1}{4} \quad \text{since} \quad 0 \le \int w \le 1.$$

So from (2.3) we get

$$\int fg - \int f \int g + \int_F |f-g| \int_G |f-g| \le \frac{1}{4}$$

and this concludes the proof of (1.3). □

Note. As we mentioned in the introduction, it is immaterial whether the left hand side of (1.2) is enclosed by modulus signs or not. However, in the case of our new inequality (1.3), although the result of doing so would be correct, it would add nothing since the left side of the modulus form, when opened, is already implied by the Grüss inequality.

3. Final Remarks

Referring back to (1.4) let us now take

$$\int f \quad \text{to mean the Riemann-Stieltjes integral} \quad \int_0^1 f(x)d\alpha, \text{ etc.,}$$

where now $f, g \in C[0,1]$, $0 \leq f(x), g(x) \leq 1$ and $\alpha(x)$ is non-decreasing from 0 to 1 in $[0,1]$.

All the calculations in the previous section proceed just as before and we arrive at a more general form of (1.3), namely:

$$\int_0^1 f(x)g(x)d\alpha - \int_0^1 f(x)d\alpha \int_0^1 g(x)d\alpha + \int_0^1 \phi_F \left|f(x) - g(x)\right| d\alpha \int_0^1 \phi_G \left|f(x) - g(x)\right| d\alpha \leq \frac{1}{4} \tag{3.1}$$

in which ϕ_F and ϕ_G are the characteristic functions of F and G. We have written the last two integrals here in this way so that all the integrands in (3.1) are seen to be continuous functions, as indeed, are the functions u and w which appear in the calculations.

An equivalent form of (3.1) is

$$L(fg) - L(f)L(g) + L(\phi_F \left|f - g\right|)L(\phi_G \left|f - g\right| \leq \frac{1}{4},$$

where L is a positive linear functional defined on $C[0,1]$

Next, suppose that in (3.1) the function α is a step function with points of increase $\frac{1}{n}$ at each x_k where $0 < x_1 < x_2 < \cdots < x_n < 1$. Then writing a_k and b_k for $f(x_k)$ and $g(x_k)$ respectively we get the discrete form of our inequality:

$$\frac{1}{n}\sum_{1}^{n} a_k b_k - \frac{1}{n}\sum_{1}^{n} a_k \cdot \frac{1}{n}\sum_{1}^{n} b_k + \frac{1}{n}\sum_{k\in F} |a_k - b_k| \cdot \frac{1}{n}\sum_{k\in G} |a_k - b_k| \le \frac{1}{4},$$

with $0 \le a_k, b_k \le 1$ and $F = \{k : a_k > b_k\}$, $G = \{k : a_k < b_k\}$.

References

[1] G. GRÜSS, Uber das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215–226.

[2] A.McD. MERCER AND P. MERCER, New proofs of the Grüss inequality. *Aust. J. Math. Anal. Appls.*, **1**(2) (2004), Art. 12.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.