

Journal of Inequalities in Pure and Applied Mathematics

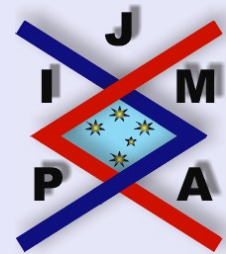
AN IMPROVEMENT OF THE GRÜSS INEQUALITY

A.McD. MERCER

Department of Mathematics and Statistics
University of Guelph
Guelph, Ontario K8N 2W1, Canada.

EMail: amercer@reach.net

©2000 Victoria University
ISSN (electronic): 1443-5756
161-05



volume 6, issue 4, article 93,
2005.

*Received 21 May, 2005;
accepted 27 July, 2005.*

Communicated by: P.S. Bullen

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

The Grüss inequality is improved by adding a positive component to the left hand side.

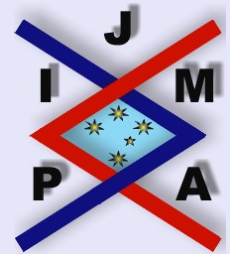
2000 Mathematics Subject Classification: 26D15.

Key words: Inequality, Stieltjes integral, Linear functional.

This note is dedicated to the memory of my wife Mari.

Contents

1	Introduction	3
2	Proof of the Theorem	6
3	Final Remarks	8
	References	



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 2 of 10

1. Introduction

Let $L^\infty(a, b)$ denote the usual Banach algebra of essentially bounded functions defined a.e. on (a, b) and let the functions f and g be members of this set with $m \leq f(x) \leq M$, $p \leq g(x) \leq P$ a.e.

Then the classical Grüss inequality [1] reads as follows:

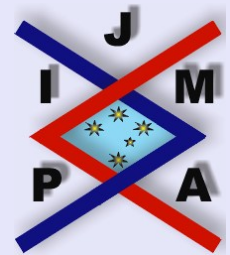
$$(1.1) \quad \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \leq \frac{1}{4} (M - m) (P - p).$$

Proofs of this inequality and other forms of it can be found, for example, in [2] and Chapter 10 of [3] serves as a comprehensive reference. There are also many references to be found at the web site <http://jipam.vu.edu.au>.

Since (1.1) is invariant under affine transformations of f and g (i.e. $f \rightarrow Af+B$, $g \rightarrow Cg+D$) we could, without any loss of generality, put $M = P = 1$ and $m = p = 0$. It may be noted also that if in (1.1) we were to replace $f(x)$ by $M + m - f(x)$ we would obtain:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \geq -\frac{1}{4} (M - m) (P - p)$$

so it is immaterial whether we enclose the left hand side of (1.1) within modulus signs or not.



An Improvement of the Grüss
Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 3 of 10

We mention this because the Grüss inequality appears in both forms. Finally, there is no loss in taking the basic interval (a, b) to be $(0, 1)$.

To sum up, the inequality stated as:

$$(1.2) \quad \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx \leq \frac{1}{4}$$

with $0 \leq f(x), g(x) \leq 1$ a.e. is entirely equivalent to (1.1).

It is our purpose in this note to show that the Grüss inequality (1.2) can be sharpened to the following:

Theorem 1.1. *We have*

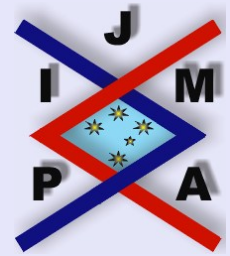
$$(1.3) \quad \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx + \int_F |f(x) - g(x)| dx \int_G |f(x) - g(x)| dx \leq \frac{1}{4},$$

where F and G are the sets $F = \{x : f(x) \geq g(x)\}$, $G = \{x : f(x) < g(x)\}$.

It is interesting to note that both (1.2) and (1.3) are best-possible in the sense that there are functions which give equality in each. These functions are:

$$f(x) = g(x) = 1 \quad \text{if} \quad 0 < x < \frac{1}{2},$$

$$f(x) = g(x) = 0 \quad \text{if} \quad \frac{1}{2} \leq x < 1.$$



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 4 of 10

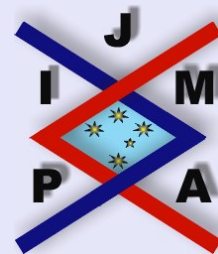
On the other hand, the Grüss inequality (1.2) is not best-possible in a more conventional sense. For, if we take

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \frac{1}{2} \quad \text{if } 0 \leq x \leq 1$$

we find that the left hand side of the Grüss inequality (1.2) is zero whereas the value of the left side of our new inequality (1.3) is $\frac{1}{16}$.

We shall write

$$(1.4) \quad \int f \text{ to mean } \int_0^1 f(x)dx \text{ etc.}$$



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 5 of 10

2. Proof of the Theorem

We now prove the result stated in (1.3).

Proof. Let $u(x) = \max[f(x), g(x)]$ and $w(x) = \min[f(x), g(x)]$.

Then $f(x)g(x) = u(x)w(x)$, so that

$$(2.1) \quad \int fg = \int uw.$$

Next

$$\begin{aligned} & \int f \int g - \int u \int w \\ &= \left[\int_F f + \int_G f \right] \left[\int_F g + \int_G g \right] - \left[\int_F f + \int_G g \right] \left[\int_F g + \int_G f \right]. \end{aligned}$$

This reduces to

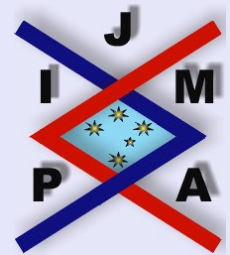
$$\left[\int_F f - \int_F g \right] \left[\int_G g - \int_G f \right],$$

which equals

$$\int_F |f - g| \int_G |f - g|$$

and so we have

$$(2.2) \quad \int f \int g - \int u \int w = \int_F |f - g| \int_G |f - g|.$$



An Improvement of the Grüss
Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 6 of 10

From (2.1) and (2.2) we get

$$(2.3) \quad \int fg - \int f \int g + \int_F |f - g| \int_G |f - g| = \int uw - \int u \int w.$$

Since $0 < f, g < 1$ then $0 < w \leq u < 1$ and so the right hand side here is majorised by

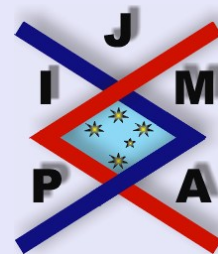
$$\int w - \left(\int w \right)^2 \leq \frac{1}{4} \quad \text{since} \quad 0 \leq \int w \leq 1.$$

So from (2.3) we get

$$\int fg - \int f \int g + \int_F |f - g| \int_G |f - g| \leq \frac{1}{4}$$

and this concludes the proof of (1.3). □

Note. As we mentioned in the introduction, it is immaterial whether the left hand side of (1.2) is enclosed by modulus signs or not. However, in the case of our new inequality (1.3), although the result of doing so would be correct, it would add nothing since the left side of the modulus form, when opened, is already implied by the Grüss inequality.



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 7 of 10

3. Final Remarks

Referring back to (1.4) let us now take

$$\int f \text{ to mean the Riemann-Stieltjes integral } \int_0^1 f(x)d\alpha, \text{ etc.,}$$

where now $f, g \in C[0, 1]$, $0 \leq f(x), g(x) \leq 1$ and $\alpha(x)$ is non-decreasing from 0 to 1 in $[0, 1]$.

All the calculations in the previous section proceed just as before and we arrive at a more general form of (1.3), namely:

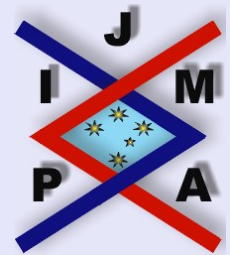
$$(3.1) \quad \int_0^1 f(x)g(x)d\alpha - \int_0^1 f(x)d\alpha \int_0^1 g(x)d\alpha + \int_0^1 \phi_F |f(x) - g(x)| d\alpha \int_0^1 \phi_G |f(x) - g(x)| d\alpha \leq \frac{1}{4}$$

in which ϕ_F and ϕ_G are the characteristic functions of F and G . We have written the last two integrals here in this way so that all the integrands in (3.1) are seen to be continuous functions, as indeed, are the functions u and w which appear in the calculations.

An equivalent form of (3.1) is

$$L(fg) - L(f)L(g) + L(\phi_F |f - g|)L(\phi_G |f - g|) \leq \frac{1}{4},$$

where L is a positive linear functional defined on $C[0, 1]$



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

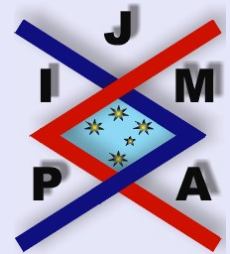
Quit

Page 8 of 10

Next, suppose that in (3.1) the function α is a step function with points of increase $\frac{1}{n}$ at each x_k where $0 < x_1 < x_2 < \cdots < x_n < 1$. Then writing a_k and b_k for $f(x_k)$ and $g(x_k)$ respectively we get the discrete form of our inequality:

$$\frac{1}{n} \sum_1^n a_k b_k - \frac{1}{n} \sum_1^n a_k \cdot \frac{1}{n} \sum_1^n b_k + \frac{1}{n} \sum_{k \in F} |a_k - b_k| \cdot \frac{1}{n} \sum_{k \in G} |a_k - b_k| \leq \frac{1}{4},$$

with $0 \leq a_k, b_k \leq 1$ and $F = \{k : a_k > b_k\}$, $G = \{k : a_k < b_k\}$.



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

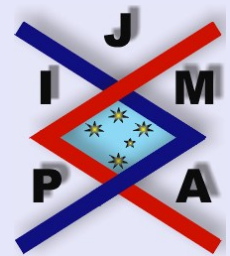
Close

Quit

Page 9 of 10

References

- [1] G. GRÜSS, Über das Maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215–226.
- [2] A.McD. MERCER AND P. MERCER, New proofs of the Grüss inequality. *Aust. J. Math. Anal. Appls.*, **1**(2) (2004), Art. 12.
- [3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.



An Improvement of the Grüss Inequality

A.McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 10 of 10