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ON A RESULT OF TOHGE CONCERNING THE UNICITY OF MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper we prove some uniqueness theorems of meromorphic functions which improve a result of Tohge and answer a question given by him. Furthermore, an example shows that the conditions of our results are sharp.

Key words and phrases: Meromorphic functions, Weighted sharing, Uniqueness.

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1. INTRODUCTION, DEFINITIONS AND RESULTS

Let $f(z)$ be a nonconstant meromorphic function in the complex plane C. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, and $m(r, f)$ (see, e.g., [1]). In this paper, we use $N_{k)}(r, 1/(f-a))$ to denote the counting function of a-points of f with multiplicities less than or equal to k, and $N_{(k}(r, 1/(f-a))$ the counting function of a-points of f with multiplicities greater than or equal to k. We also use $\overline{N}_{k)}(r, 1/(f-a))$ and $\overline{N}_{(k}(r, 1/(f-a))$ to denote the corresponding reduced counting functions, respectively (see [2]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure.

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Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and a be a complex number. If the zeros of $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities), then we say that f and g share the value a CM (IM).

Let $S_0(f=a=g)$ be the set of all common zeros of $f(z)-a$ and $g(z)-a$ ignoring multiplicities, $S_E(f=a=g)$ be the set of all common zeros of $f(z)-a$ and $g(z)-a$ with the same multiplicities. Denote by $\overline{N}_0(r,f=a=g)$, $\overline{N}_E(r,f=a=g)$ the reduced counting functions of f and g corresponding to the sets $S_0(f=a=g)$ and $S_E(f=a=g)$, respectively. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_0(r,f=a=g)=S(r,f)+S(r,g),$$

then we say that f and g share a IM*. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,f=a=g)=S(r,f)+S(r,g),$$

then we say that f and g share a CM*.

Let k be a positive integer or infinity. We denote by $\overline{E}_{k)}(a,f)$ the set of a-points of f with multiplicities less than or equal to k (ignoring multiplicities).

In 1988, Tohge [3] proved the following result.

Theorem A ([3]). *Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM, and f', g' share 0 CM. Then f and g satisfy one of the following relations:*

(i) $f\equiv g$,
(ii) $fg\equiv 1$,
(iii) $(f-1)(g-1)\equiv 1$,
(iv) $f+g\equiv 1$,
(v) $f\equiv cg$,
(vi) $f-1\equiv c(g-1)$,
(vii) $[(c-1)f+1][(c-1)g-c]\equiv -c$,

where c $(\neq 0,1)$ is a constant.

In the same paper, Tohge [3] suggested the following problem: *Is it possible to weaken the restriction of CM sharing in Theorem A?*

In 2000, Al-Khaladi [4] – [5] dealt with this problem and proved the following theorems, which are improvements of Theorem A.

Theorem B ([4]). *Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM, and f', g' share 0 IM. Then the conclusions of Theorem A still hold.*

Theorem C ([5]). *Let f and g be two nonconstant meromorphic functions sharing $0, \infty$ CM, and f', g' share 0 IM. If $\overline{E}_{k)}(1,f)=\overline{E}_{k)}(1,g)$, where k is a positive integer or infinity, then the conclusions of Theorem A still hold.*

Now we explain the notion of weighted sharing as introduced in [6] – [7].

Definition 1.1 ([6] – [7]). Let k be a nonnegative integer or infinity. For $a\in C\bigcup\{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a,f)=E_k(a,g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\leq k)$ and z_0 is a zero of $f-a$ with multiplicity m $(>k)$ if and only if it is a zero of $g-a$ with multiplicity n $(>k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In particular, if f, g share a value a IM* or CM*, then we say that f, g share $(a, 0)^*$ or $(a, \infty)^*$ respectively (see [8]).

Definition 1.2 ([8])**.** For $a \in C \bigcup \{\infty\}$, we put

$$\delta_{(p}(a, f) = 1 - \limsup_{r \to \infty} \frac{N_{(p}\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

where p is a positive number.

In 2005, the present author etc. [8] and Lahiri [9] also improved Theorem A and obtained the following results, respectively.

Theorem D ([8])**.** *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, $(1, \infty)$, (∞, ∞), and f', g' share $(0, 0)^*$. If $\delta_{(2}(0, f) > 1/2$, then the conclusions of Theorem A still hold.*

Theorem E ([9])**.** *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, $(1, m)$, and (∞, k), where k, m are positive integers or infinities satisfying $(m-1)(km-1) > (1+m)^2$. If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then the conclusions of Theorem A still hold.*

In this paper, we shall prove the following theorems, which improve and supplement the above theorems.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, k_3), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_j $(j = 1, 2, 3)$ are positive integers satisfying*

$$k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2. \tag{1.1}$$

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f - 1)(g - 1) \equiv 1$,
(iv) $f + g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f - 1 \equiv c(g - 1)$,
(vii) $[(c - 1)f + 1][(c - 1)g - c] \equiv -c$,
where c $(\neq 0, 1)$ is a constant.

From Theorem 1.1, we immediately deduce the following corollary.

Corollary 1.2. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, k_3), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_j $(j = 1, 2, 3)$ are positive integers satisfying one of the following relations:*

(i) $k_1 \geq 1$, $k_2 \geq 3$, *and* $k_3 \geq 4$,
(ii) $k_1 \geq 2$, $k_2 \geq 2$, *and* $k_3 \geq 3$,
(iii) $k_1 \geq 1$, $k_2 \geq 2$, *and* $k_3 \geq 6$.

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f+g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f-1 \equiv c(g-1)$,
(vii) $[(c-1)f+1][(c-1)g-c] \equiv -c$,

where $c\ (\neq 0,1)$ is a constant.

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and $k_j\ (j=1,2,3)$ are positive integers satisfying (1.1). If*

$$N_{1)}\left(r,\frac{1}{f'}\right)+N_{1)}\left(r,\frac{1}{g'}\right)<(\lambda+o(1))T(r),\quad (r\in I), \tag{1.2}$$

where $0<\lambda<1/3$, $T(r)=\max\{T(r,f),T(r,g)\}$, and I is a set of infinite linear measure, then f and g satisfy one of the following relations: (i)$f\equiv g$, (ii) $fg\equiv 1$, (iii)$(f-1)(g-1)\equiv 1$, (iv)$f+g\equiv 1$, (v)$f\equiv cg$, (vi)$f-1\equiv c(g-1)$, (vii) $[(c-1)f+1][(c-1)g-c]\equiv -c$, where c $(\neq 0,1)$ is a constant.

By Theorem 1.3, we instantly derive the following corollary.

Corollary 1.4. *Let f and g be two nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and $k_j\ (j=1,2,3)$ are positive integers satisfying one of the following relations:*

(i) $k_1\geq 1$, $k_2\geq 3$, *and* $k_3\geq 4$,
(ii) $k_1\geq 2$, $k_2\geq 2$, *and* $k_3\geq 3$,
(iii) $k_1\geq 1$, $k_2\geq 2$, *and* $k_3\geq 6$.

If (1.2) holds, then f and g satisfy one of the following relations:

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f+g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f-1 \equiv c(g-1)$,
(vii) $[(c-1)f+1][(c-1)g-c] \equiv -c$,

where $c\ (\neq 0,1)$ is a constant.

The following example shows that any one of $k_j\ (j=1,2,3)$ in Theorem 1.1, Corollary 1.2, Theorem 1.3 and Corollary 1.4 cannot be equal to 0.

Example 1.1. Let $f=(e^z-1)^{-2}$ and $g=(e^z-1)^{-1}$. Then f and g share $(0,\infty)$, $(1,\infty)$, $(\infty,0)$, and f', g' share $(0,\infty)$. However, f and g do not satisfy any one of the relations given in Theorem 1.1, Corollary 1.2, Theorem 1.3 and Corollary 1.4.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([10]). *Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(1,0)$, and $(\infty,0)$. Then*

$$T(r,f)\leq 3T(r,g)+S(r,f),\qquad T(r,g)\leq 3T(r,f)+S(r,g),$$

$$S(r,f)=S(r,g):=S(r).$$

Proof. Note that f and g share $(0,0)$, $(1,0)$, and $(\infty,0)$. By the second fundamental theorem, we can easily obtain the conclusion of Lemma 2.1. □

The second lemma is due to Yi [11], which plays an important role in the proof.

Lemma 2.2 ([11]). *Let f and g be two distinct nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and k_j $(j=1,2,3)$ are positive integers satisfying (1.1). Then*

$$\overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)=S(r),$$

the same identity holds for g.

Lemma 2.3. *Let f and g be two nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and k_j $(j=1,2,3)$ are positive integers satisfying (1.1). If*

$$\alpha=\frac{g}{f}, \tag{2.1}$$

$$\beta=\frac{f-1}{g-1}, \tag{2.2}$$

then

$$\overline{N}\left(r,\frac{1}{\alpha}\right)=\overline{N}(r,\alpha)=\overline{N}\left(r,\frac{1}{\beta}\right)=\overline{N}(r,\beta)=S(r).$$

Proof. If α or β is a constant, then the result is obvious. Next we suppose that α and β are nonconstant. Since f and g share (a_1,k_1), (a_2,k_2), and (a_3,k_3), by (2.1), (2.2), and Lemma 2.2 we have

$$\overline{N}\left(r,\frac{1}{\alpha}\right)\le\overline{N}_{(2}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}(r,f)=S(r),$$

$$\overline{N}(r,\alpha)\le\overline{N}_{(2}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}(r,g)=S(r),$$

$$\overline{N}(r,\frac{1}{\beta})\le\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(2}(r,g)=S(r),$$

$$\overline{N}(r,\beta)\le\overline{N}_{(2}\left(r,\frac{1}{g-1}\right)+\overline{N}_{(2}(r,f)=S(r),$$

which completes the proof of the lemma. □

Lemma 2.4. *Let f and g be two distinct nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and k_j $(j=1,2,3)$ are positive integers satisfying (1.1). If f is not a fractional linear transformation of g, then*

$$\overline{N}_{(2}\left(r,\frac{1}{f'}\right)=S(r),\qquad\overline{N}_{(2}\left(r,\frac{1}{g'}\right)=S(r).$$

Proof. Without loss of generality, we assume that $a_1=0$, $a_2=1$, and $a_3=\infty$. Let α and β be given by (2.1) and (2.2). From (2.1) and (2.2), we have

$$f=\frac{1-\beta}{1-\alpha\beta}, \tag{2.3}$$

$$g = \frac{(1-\beta)\alpha}{1-\alpha\beta}. \tag{2.4}$$

Since f is not a fractional linear transformation of g, we know that α, β, and $\alpha\beta$ are nonconstant. Let

$$h := \frac{\alpha\beta'}{\alpha\beta' + \alpha'\beta} = \frac{\beta'/\beta}{\alpha'/\alpha + \beta'/\beta}. \tag{2.5}$$

Then we have $h \not\equiv 0, 1$. Note that

$$N\left(r, \frac{\alpha'}{\alpha}\right) = \overline{N}\left(r, \frac{1}{\alpha}\right) + \overline{N}(r, \alpha),$$
$$N\left(r, \frac{\beta'}{\beta}\right) = \overline{N}\left(r, \frac{1}{\beta}\right) + \overline{N}(r, \beta).$$

From this and Lemma 2.3, we get

$$T\left(r, \frac{\alpha'}{\alpha}\right) = T\left(r, \frac{\beta'}{\beta}\right) = S(r), \tag{2.6}$$

and so

$$T(r, h) = S(r). \tag{2.7}$$

By (2.3), we get

$$f - h = \frac{(1-\beta) - h(1-\alpha\beta)}{1-\alpha\beta}. \tag{2.8}$$

Let

$$F := (f-h)(1-\alpha\beta) = (1-\beta) - h(1-\alpha\beta). \tag{2.9}$$

From (2.5) and (2.9), we have

$$\frac{F'}{F} - \frac{\beta'}{\beta} = \frac{-\beta' - h'(1-\alpha\beta) + \alpha\beta' - \beta' F/\beta}{F} = \frac{1}{f-h}\left[\frac{\beta'}{\beta}(h-1) - h'\right]. \tag{2.10}$$

If $\beta'(h-1)/\beta - h' \equiv 0$, then from this and (2.10), we get

$$h = c_1\beta + 1, \tag{2.11}$$

and so $F'/F - \beta'/\beta \equiv 0$, i.e.,

$$F = c_2\beta, \tag{2.12}$$

where c_1, c_2 are nonzero constants. By (2.7), (2.11), and (2.12), we have

$$T(r, F) = T(r, \beta) = S(r).$$

From this, (2.7), and (2.9), we get

$$T(r, \alpha) = S(r),$$

and so $T(r, f) = S(r)$, which is impossible. Therefore $\beta'(h-1)/\beta - h' \not\equiv 0$. By (2.10), we have

$$\frac{1}{f-h} = \frac{F'/F - \beta'/\beta}{\beta'(h-1)/\beta - h'}. \tag{2.13}$$

From (2.6), (2.7), and (2.13), we get

$$m\left(r, \frac{1}{f-h}\right) \le m\left(r, \frac{F'}{F}\right) + S(r) = S(r). \tag{2.14}$$

Since F'/F and β'/β have only simple poles, it follows again from (2.6), (2.7), and (2.13) that

$$\begin{aligned} N_{(2}\left(r, \frac{1}{f-h}\right) &\le 2N\left(r, \frac{1}{\beta'(h-1)/\beta - h'}\right) + S(r) \\ &\le 2T\left(r, \frac{\beta'(h-1)}{\beta} - h'\right) + S(r) \\ &\le 2T\left(r, \frac{\beta'}{\beta}\right) + 2T(r,h) + 2T(r,h') + S(r) \\ &\le S(r), \end{aligned}$$

i.e.,

$$N_{(2}\left(r, \frac{1}{f-h}\right) = S(r). \tag{2.15}$$

By (2.2) and (2.4), we have

$$\frac{g-f}{g-1} = 1-\beta,$$

$$\frac{g'}{g} = \frac{\alpha'(1-\alpha\beta) + (\alpha-1)(\alpha\beta' + \alpha'\beta)}{\alpha(1-\beta)(1-\alpha\beta)}.$$

Therefore

$$\frac{g'(g-f)}{g(g-1)} = \frac{(1-\beta)(\alpha\beta' + \alpha'\beta) - \alpha\beta'(1-\alpha\beta)}{\alpha\beta(1-\alpha\beta)}. \tag{2.16}$$

From (2.5) and (2.8), we get

$$(f-h)\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right) = \frac{(1-\beta)(\alpha\beta' + \alpha'\beta) - \alpha\beta'(1-\alpha\beta)}{\alpha\beta(1-\alpha\beta)}. \tag{2.17}$$

By (2.16) and (2.17), we have

$$\frac{g'(g-f)}{g(g-1)} = (f-h)\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right). \tag{2.18}$$

Let $N_0^{(2}(r, 1/g')$ denote the counting function corresponding to multiple zeros of g' that are not zeros of g and $g-1$. Then from (2.15) and (2.18), we get

$$N_0^{(2}\left(r, \frac{1}{g'}\right) \le N_{(2}\left(r, \frac{1}{f-h}\right) + S(r) \le S(r).$$

From this and Lemma 2.2, we have

$$\overline{N}_{(2}\left(r, \frac{1}{g'}\right) \le N_0^{(2}\left(r, \frac{1}{g'}\right) + \overline{N}_{(2}\left(r, \frac{1}{g}\right) + \overline{N}_{(2}\left(r, \frac{1}{g-1}\right) \le S(r),$$

i.e.,

$$\overline{N}_{(2}\left(r, \frac{1}{g'}\right) = S(r).$$

Similarly, we can prove

$$\overline{N}_{(2}\left(r, \frac{1}{f'}\right) = S(r),$$

which also completes the proof of Lemma 2.4. □

Lemma 2.5. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, k_3), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_j $(j = 1, 2, 3)$ are positive integers satisfying* (1.1)*. If f is a fractional linear transformation of g, then f and g satisfy one of the following relations:*

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f + g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f - 1 \equiv c(g-1)$,
(vii) $[(c-1)f+1][(c-1)g-c] \equiv -c$,

where c $(\neq 0, 1)$ is a constant.

Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Since f is a fractional linear transformation of g, we can suppose that

$$f = \frac{Ag + B}{Cg + D},$$

where A, B, C, D are constants such that $AD - BC \neq 0$.

If $f \equiv g$, then the relation (i) holds. Next we assume that $f \not\equiv g$ and discuss the following cases.

Case 1 If none of 0, 1, and ∞ are Picard's exceptional values of f and g, then $f \equiv g$, which contradicts the assumption.

Case 2 If 0 and 1 are all Picard's exceptional values of f and g, then $f = \alpha g + \beta = \alpha(g + \beta/\alpha)$, where α $(\neq 0)$, β are constants. Since $f \neq 0$, it follows that $\beta/\alpha = 0$ or -1.

Subcase 2.1 If $\beta = 0$, then $f = \alpha g$, i.e., $f - 1 = \alpha(g - 1/\alpha)$. Since $f \neq 1$, it follows that $\alpha = 1$ and so $f \equiv g$. This is a contradiction.

Subcase 2.2 If $\beta/\alpha = -1$, then $f = \alpha g - \alpha$, i.e., $f - 1 = \alpha(g - (\alpha+1)/\alpha)$. Since $f \neq 1$, it follows that $\alpha = -1$. Thus $f \equiv -g + 1$, which implies the relation (iv).

Case 3 If 1 and ∞ are all Picard's exceptional values of f and g, then $f = Ag/(Cg + D)$, where A $(\neq 0)$, D $(\neq 0)$ are constants.

Subcase 3.1 If $C = 0$, then $f = \alpha g$, i.e., $f - 1 = \alpha(g - 1/\alpha)$, where α $(\neq 0)$ is a constant. Since $f \neq 1$ and $g \neq 1, \infty$, it follows that $\alpha = 1$ and so $f \equiv g$. This is a contradiction.

Subcase 3.2 If $C \neq 0$, then $f = \alpha g/(g-1)$, i.e., $f - 1 = ((\alpha - 1)g + 1)/(g - 1)$, where α $(\neq 0)$ is a constant. Since $f \neq 1$ and $g \neq 1, \infty$, it follows that $\alpha = 1$ and so $f - 1 \equiv 1/(g-1)$. This is the relation (iii).

Case 4 If 0 and ∞ are all Picard's exceptional values of f and g, then $f = (Ag+B)/(Cg+D)$, where $A + B = C + D$.

Subcase 4.1 If $A = 0$, then $f = B/(Cg + D)$, where B $(\neq 0)$, C $(\neq 0)$ are constants. Since $f \neq \infty$ and $g \neq 0, \infty$, it follows that $D = 0$. Thus $fg \equiv 1$ because f and g share $(1, k_2)$. This is the relation (ii).

Subcase 4.2 If $A \neq 0$ and $C = 0$, then $f = \alpha g + \beta$, where α $(\neq 0)$, β are constants. Since $f \neq 0$ and $g \neq 0, \infty$, it follows that $\beta = 0$. Thus $f \equiv g$ because f and g share $(1, k_2)$. This is a contradiction.

Subcase 4.3 If $A \neq 0$ and $C \neq 0$, then it follows that $B = D = 0$ because $f \neq 0, \infty$ and $g \neq 0, \infty$. Thus $f \equiv$ constant, which contradicts the assumption.

Case 5 If 0 is Picard's exceptional value of f and g but 1 and ∞ are not, then it follows that $C = 0$ because f and g share (∞, k_3). Thus $f = \alpha g + \beta$, where α $(\neq 0)$, β are constants such that $\alpha + \beta = 1$.

Subcase 5.1 If $\beta = 0$, then it follows that $\alpha = 1$ and so $f \equiv g$. This is a contradiction.
Subcase 5.2 If $\beta \neq 0$, then it follows that $\beta = 1 - \alpha$ and so $f \equiv \alpha g + 1 - \alpha$, where α $(\neq 0, 1)$ is a constant. This is the relation (vi).
Case 6 If 1 is Picard's exceptional value of f and g but 0 and ∞ are not, then it follows that $C = 0$ because f and g share (∞, k_3). Since f and g share $(0, k_1)$, it follows that $B = 0$ and so $f \equiv \alpha g$, where α $(\neq 0)$ is a constant. If $\alpha = 1$, then $f \equiv g$, which is a contradiction. Thus $f \equiv \alpha g$, where α $(\neq 0, 1)$ is a constant. This is the relation (v).
Case 7 If ∞ is Picard's exceptional value of f and g but 0 and 1 are not, then it follows that $B = 0$ and $A = C + D$ because f and g share $(0, k_1)$ and $(1, k_2)$. Thus $f = Ag/(Cg + D)$, where A $(\neq 0)$, D $(\neq 0)$ are constants.
Subcase 7.1 If $C = 0$, then it follows that $A = D$ because f and g share $(1, k_2)$. Thus $f \equiv g$, which is a contradiction.
Subcase 7.2 If $C \neq 0$, then it follows that $f = \alpha g/(g + \beta)$ and $\alpha = 1 + \beta$, where α $(\neq 0, 1)$, β are constants. Thus $f \equiv \alpha g/(g + \alpha - 1)$, i.e., $fg - (1 - \alpha)f - \alpha g \equiv 0$, which implies the relation (vii).

This completes the proof of Lemma 2.5. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Otherwise, a fractional linear transformation will do. Let α and β be given by (2.1) and (2.2).

Suppose now that f is not a fractional linear transformation of g. Then from Lemma 2.4, we have

$$\overline{N}_{(2}\left(r, \frac{1}{f'}\right) = S(r), \qquad \overline{N}_{(2}\left(r, \frac{1}{g'}\right) = S(r). \tag{3.1}$$

By (2.1), we get

$$\frac{\alpha'}{\alpha} = \frac{g'}{g} - \frac{f'}{f},$$

i.e.,

$$\frac{\alpha'}{\alpha} f = \frac{f}{g} g' - f'. \tag{3.2}$$

Let z_0 be a simple zero of g' that is not a zero of f and g. Then it follows that z_0 is a simple zero of f' because $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$. Again from (3.2), we deduce that z_0 is a zero of α'/α. On the other hand, the process of proving Lemma 2.4 shows that

$$T\left(r, \frac{\alpha'}{\alpha}\right) = T\left(r, \frac{\beta'}{\beta}\right) = S(r).$$

From this, (3.1), and Lemma 2.2, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{g'}\right) &= \overline{N}_{(2}\left(r, \frac{1}{g'}\right) + N_{1)}\left(r, \frac{1}{g'}\right) \\ &\leq N\left(r, \frac{\alpha'}{\alpha}\right) + \overline{N}_{(2}\left(r, \frac{1}{g}\right) + S(r) \\ &\leq S(r). \end{aligned} \tag{3.3}$$

Similarly, we can prove

$$\overline{N}\left(r, \frac{1}{f'}\right) = S(r). \tag{3.4}$$

Let

$$\Delta_1 := \left(\frac{f''}{f'} - \frac{2f'}{f}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g}\right).$$

If $\Delta_1 \equiv 0$, then by integration we obtain

$$\frac{1}{f} = \frac{c}{g} + d,$$

i.e.,

$$f = \frac{g}{c + dg},$$

where $c\ (\neq 0)$, d are constants. Thus f is a fractional linear transformation of g, which contradicts the assumption. Hence $\Delta_1 \not\equiv 0$.

Since f and g share $(0, k_1)$, it follows that a simple zero of f is a simple zero of g and conversely. Let z_0 be a simple zero of f and g. Then in some neighborhood of z_0, we get $\Delta_1 = (z - z_0)\gamma(z)$, where γ is analytic at z_0. Thus by (3.3), (3.4), and Lemma 2.2, we get

$$\begin{aligned}
N_{1)}\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{1}{\Delta_1}\right) \\
&\leq N(r, \Delta_1) + S(r) \\
&\leq \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + \overline{N}_{(2}\left(r, \frac{1}{f}\right) \\
&\qquad + \overline{N}_{(2}\left(r, \frac{1}{g}\right) + \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + S(r) \\
&\leq S(r),
\end{aligned}$$

and so

$$\overline{N}\left(r, \frac{1}{f}\right) = N_{1)}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f}\right) = S(r). \tag{3.5}$$

Let

$$\Delta_2 := \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right),$$

and

$$\Delta_3 := \frac{f''}{f'} - \frac{g''}{g'}.$$

In the same manner as the above, we can obtain

$$\overline{N}\left(r, \frac{1}{f-1}\right) = S(r), \tag{3.6}$$

and

$$\overline{N}(r, f) = S(r). \tag{3.7}$$

From (3.5), (3.6), (3.7), and the second fundamental theorem, we have

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r) \leq S(r),$$

which is a contradiction. Therefore f is a fractional linear transformation of g. Again from Lemma 2.5, we obtain the conclusion of Theorem 1.1. □

Proof of Theorem 1.3. Likewise, we can assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Suppose now that f is not a fractional linear transformation of g.

Let

$$T(r) = \begin{cases} T(r,f), & \text{for} \quad r \in I_1, \\ T(r,g), & \text{for} \quad r \in I_2, \end{cases} \tag{3.8}$$

where

$$I = I_1 \cup I_2. \tag{3.9}$$

Note that I is a set of infinite linear measure of $(0, \infty)$. We can see by (3.9) that I_1 is a set of infinite linear measure of $(0, \infty)$ or I_2 is a set of infinite linear measure of $(0, \infty)$. Without loss of generality, we assume that I_1 is a set of infinite linear measure of $(0, \infty)$. Then by (3.8), we have

$$T(r) = T(r,f). \tag{3.10}$$

Let Δ_1, Δ_2, and Δ_3 be defined as in Theorem 1.1. Similar to the proof of (3.5), (3.6), and (3.7) in Theorem 1.1, we easily get

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f}\right) &= N_{1)}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f}\right) \\ &\leq N_{1)}\left(r, \frac{1}{f'}\right) + N_{1)}\left(r, \frac{1}{g'}\right) + S(r), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f-1}\right) &= N_{1)}\left(r, \frac{1}{f-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) \\ &\leq N_{1)}\left(r, \frac{1}{f'}\right) + N_{1)}\left(r, \frac{1}{g'}\right) + S(r), \end{aligned} \tag{3.12}$$

and

$$\overline{N}(r,f) = N_{1)}(r,f) + \overline{N}_{(2}(r,f) \leq N_{1)}\left(r, \frac{1}{f'}\right) + N_{1)}\left(r, \frac{1}{g'}\right) + S(r). \tag{3.13}$$

From (1.2), (3.10), (3.11), (3.12), (3.13), and the second fundamental theorem, we have for $r \in I$

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r) \\ &\leq 3\left[N_{1)}\left(r, \frac{1}{f'}\right) + N_{1)}\left(r, \frac{1}{g'}\right)\right] + S(r) \\ &< 3(\lambda + o(1))T(r,f), \end{aligned}$$

which is impossible since $0 < \lambda < 1/3$. Therefore f is a fractional linear transformation of g. Again from Lemma 2.5, we obtain the conclusion of Theorem 1.3. □

4. FINAL REMARKS

Clearly, if k_j $(j = 1, 2, 3)$ are positive integers satisfying (1.1), then

$$k_j k_i > 1 \quad (j \neq i, j, i = 1, 2, 3).$$

Theorem 4.1. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_1 and k_2 are positive integers satisfying*

$$k_1 k_2 > 1. \tag{4.1}$$

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f + g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f - 1 \equiv c(g-1)$,
(vii) $[(c-1)f + 1][(c-1)g - c] \equiv -c$,

where $c\ (\neq 0, 1)$ is a constant.

Theorem 4.2. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k), (a_2, ∞), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k is an integer satisfying*

$$k \geq 1. \tag{4.2}$$

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f + g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f - 1 \equiv c(g-1)$,
(vii) $[(c-1)f + 1][(c-1)g - c] \equiv -c$,

where $c\ (\neq 0, 1)$ is a constant.

Theorem 4.3. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_1 and k_2 are positive integers satisfying (4.1). If (1.2) holds, then f and g satisfy one of the following relations:*

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f + g \equiv 1$,
(v) $f \equiv cg$,
(vi) $f - 1 \equiv c(g-1)$,
(vii) $[(c-1)f + 1][(c-1)g - c] \equiv -c$,

where $c\ (\neq 0, 1)$ is a constant.

Theorem 4.4. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k), (a_2, ∞), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k is an integer satisfying (4.2). If (1.2) holds, then f and g satisfy one of the following relations:*

(i) $f \equiv g$,
(ii) $fg \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$,
(iv) $f + g \equiv 1$,
(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g-1)$,
(vii) $[(c-1)f+1][(c-1)g-c] \equiv -c$,
where $c\ (\neq 0, 1)$ *is a constant.*

Proofs of Theorems 4.1 and 4.3. Without loss of generality, we assume that $k_1 \leq k_2$. Then by (4.1) we see that $k_1 \geq 1$ and $k_2 \geq 2$. Note that if f and g share (a, k) then f and g share (a, p) for all integers p, $0 \leq p < k$. Since f and g share (a_1, k_1), (a_2, k_2), and (a_3, ∞), it follows that f and g share $(a_1, 1)$, $(a_2, 2)$, and $(a_3, 6)$. Thus form Corollaries 1.2 and 1.4 we immediately obtain the conclusions of Theorems 4.1 and 4.3 respectively. □

Proofs of Theorems 4.2 and 4.4. Note that if f and g share (a_1, k), (a_2, ∞), (a_3, ∞), and $k \geq 1$, then we know that f and g share $(a_1, 1)$, $(a_2, 2)$, and $(a_3, 6)$. Thus from Corollaries 1.2 and 1.4 we instantly get the conclusions of Theorems 4.2 and 4.4 respectively. □

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