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ON SOME COVARIANCE INEQUALITIES FOR MONOTONIC AND NON-MONOTONIC FUNCTIONS

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ABSTRACT. Chebyshev's integral inequality, also known as the covariance inequality, is an important problem in economics, finance, and decision making. In this paper we derive some covariance inequalities for monotonic and non-monotonic functions. The results developed in our paper could be useful in many applications in economics, finance, and decision making.

Key words and phrases: Covariance, Chebyshev's inequality, Decisions under risk.

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1. INTRODUCTION

Chebyshev's integral inequality is widely used in applied mathematics in areas such as: economics, finance, and decision making under risk, see, for example, Wagener [8] and Athey [1]. It can also be used to study the covariance sign of two monotonic functions, see Mitrinovic, Pečarić and Fink [6] and Wagener [8].

However, monotonicity is a very strong assumption that can only sometimes be satisfied. Cuadras in [2] gave a general identity for the covariance between functions of two random variables in terms of their cumulative distribution functions. In this paper, using the Cuadras identity, we derive some integral inequalities for monotonic functions and some for non-monotonic functions.

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2. THEORY

We first present Chebyshev's algebraic inequality, see, for example, Mitrinovic, Pečarić and Fink [6], as follows:

Proposition 2.1. *Let $\alpha, \beta : [a,b] \to \mathbb{R}$ and $f(x) : [a,b] \to \mathbb{R}_+$, where $\mathbb{R}$ is the set of real numbers. We have*

(1) *if α and β are both increasing or both decreasing, then*

$$\int_a^b f(x) \int_a^b \alpha(x)\beta(x)f(x)\,dx \geq \int_a^b \alpha(x)f(x)\,dx \times \int_a^b \beta(x)f(x)\,dx\,; \tag{2.1}$$

(2) *if one is increasing and the other is decreasing, then the inequality is reversed.*

We note that in Proposition 2.1, if $f(x)$ is a probability density function, then Chebyshev's algebraic inequality in (2.1) becomes

$$Cov\big[\alpha(X), \beta(X)\big] \geq 0\,.$$

Cuadras [2] extended the work of Hoeffding [3], Mardia [4], Sen [7], and Lehmann [5] by proving that for any two real functions of bounded variation $\alpha(x)$ and $\beta(x)$ defined on $[a,b]$ and $[c,d]$, respectively, and for any two random variables X and Y such that $E\big[\,|\alpha(X)\beta(Y)|\,\big]$, $E\big[\,|\alpha(X)|\,\big]$, and $E\big[\,|\beta(Y)|\,\big]$ are finite,

$$Cov\big[\alpha(X), \beta(Y)\big] = \int_c^d \int_a^b [H(x,y) - F(x)G(y)]\,d\alpha(y)\,d\beta(x)\,, \tag{2.2}$$

where $H(x,y)$ is the joint cumulative distribution function for X and Y, and F and G are the corresponding cumulative distribution functions of X and Y, respectively.

As we noted before, the monotonicity of both functions $\alpha(X)$ and $\beta(X)$ in Proposition 2.1 is a very strong assumption, and thus, this condition may be satisfied in some situations but it could be violated in others. Thus, it is our objective in this paper to derive covariance inequalities for both monotonic functions and non-monotonic functions. We first apply the Cuadras identity to relax the monotonicity assumption of $\beta(x)$ for a single random variable in the Chebyshev inequality, as shown in the following theorem:

Theorem 2.2. *Let X be a random variable symmetric about zero with support on $[-b,b]$. Consider two real functions $\alpha(x)$ and $\beta(x)$. Assume that $\beta(x)$ is an odd function of bounded variation with $\beta(x) \geq (\leq) 0$ for all $x \geq 0$. We have*

(1) *if $\alpha(x)$ is increasing, then $Cov\big[\alpha(X), \beta(X)\big] \geq (\leq) 0$; and*
(2) *if $\alpha(x)$ is decreasing, then $Cov\big[\alpha(X), \beta(X)\big] \leq (\geq) 0$.*

Proof. We only prove Part (a) of Theorem 2.2 with $\beta(x) \geq 0$ for all $x \geq 0$. Using Cuadras' [2] identity, we obtain

$$Cov\big[\alpha(X), \beta(Y)\big] = \int_{-b}^b \int_{-b}^b \big[H(x,y) - F(x)G(y)\big] d\alpha(y)\,d\beta(x)\,, \tag{2.3}$$

where $H(x,y)$, F, and G are defined in (2.2). Since $X = Y$ in the theorem, we have $H(x,y) = F(\min\{x,y\})$. Therefore, we can write:

$$\begin{aligned} (2.4)\quad & Cov\big[\alpha(X), \beta(X)\big] \\ &= \int_{-b}^b \int_{-b}^b F[\min(x,y)]\,d\alpha(y)\,d\beta(x) - \int_{-b}^b \int_{-b}^b F(x)F(y)\,d\alpha(y)\,d\beta(x)\,. \end{aligned}$$

The second term on the right hand side of (2.4) can be expressed as

$$\begin{aligned}
&\int_{-b}^{b}\int_{-b}^{b} F(x)F(y)\,d\alpha(y)\,d\beta(x)\\
&\qquad = \int_{-b}^{b} F(y)\left[\int_{-b}^{b} F(x)\,d\beta(x)\right] d\alpha(y)\\
&\qquad = \int_{-b}^{b} F(y)\left[-\int_{-b}^{b} \beta(x)dF(x)+\beta(b)F(b)-\beta(-b)F(-b)\right] d\alpha(y)\\
&\qquad = \int_{-b}^{b} F(y)\beta(b)\,d\alpha(y) = \beta\left(b\right)\left[-\int_{-b}^{b}\alpha(y)dF(y)+\alpha(b)\right]\\
&\qquad = \beta\left(b\right)\left[\alpha\left(b\right)-\mu_{\alpha}\right], \qquad (2.5)
\end{aligned}$$

where

$$\mu_{\alpha} = \int_{-b}^{b}\alpha(y)dF(y)\,.$$

On the other hand, the first term on the right side of (2.4) becomes

$$\int_{-b}^{b}\left[\int_{-b}^{b} F[\min(x,y)]\,d\beta(x)\right] d\alpha\left(y\right) = \int_{-b}^{b}\left[\int_{-b}^{y} F(x)\,d\beta(x)+\int_{y}^{b} F(y)\,d\beta(x)\right] d\alpha\left(y\right)\ .$$

In addition, we have

$$\int_{y}^{b} F(y)\,d\beta(x) = F(y)\big[\beta(b)-\beta\left(y\right)\big]\,,$$

and hence,

$$\begin{aligned}
&\int_{-b}^{b} F(y)\big[\beta(b)-\beta\left(y\right)\big]d\alpha(y)\\
&\qquad = \int_{-b}^{b} F(y)\beta(b)\,d\alpha(y) - \int_{-b}^{b} F(y)\beta\left(y\right)d\alpha(y)\\
&\qquad = \beta(b)\big[-\mu_{\alpha}+\alpha\left(b\right)\big] - \int_{-b}^{b} F(y)\beta\left(y\right)d\alpha(y)\,.
\end{aligned}$$

Similarly, one can easily show that

$$\int_{-b}^{y} F(x)\,d\beta(x) = -\int_{-b}^{y}\beta(x)dF(x)+F(y)\beta(y)\,.$$

Thus, we have

$$\begin{aligned}
\int_{-b}^{b}\left[\int_{-b}^{y} F(x)\,d\beta(x)\right] d\alpha(y) &= \int_{-b}^{b}\left[-\int_{-b}^{y}\beta(x)dF(x)+F(y)\beta(y)\right] d\alpha(y)\\
&= -\int_{-b}^{b}\left[\int_{-b}^{y}\beta(x)dF(x)\right] d\alpha(y) + \int_{-b}^{b} F(y)\beta(y)\,d\alpha(y)\,,
\end{aligned}$$

and hence,

$$\int_{-b}^{b}\left[\int_{-b}^{b}F[\min(x,y)]\,d\beta(x)\right]d\alpha\left(x\right)$$
$$=\beta(b)\big[-\mu_{\alpha}+\alpha\left(b\right)\big]-\int_{-b}^{b}F(y)\beta\left(y\right)d\alpha(y)$$
$$-\int_{-b}^{b}\left[\int_{-b}^{y}\beta(x)dF(x)\right]d\alpha(y)+\int_{-b}^{b}F(y)\beta(y)\,d\alpha(y)$$
$$=\beta(b)\big[-\mu_{\alpha}+\alpha\left(b\right)\big]-\int_{-b}^{b}\left[\int_{-b}^{y}\beta(x)dF(x)\right]d\alpha(y)\,. \tag{2.6}$$

Thereafter, substituting (2.5) and (2.6) into (2.4), we get:

$$Cov\big[\alpha(X),\beta(X)\big]$$
$$=\beta(b)\big[-\mu_{\alpha}+\alpha\left(b\right)\big]-\int_{-b}^{b}\left[\int_{-b}^{y}\beta(x)dF(x)\right]d\alpha(y)-\beta(b)\big[-\mu_{\alpha}+\alpha\left(b\right)\big]$$
$$=-\int_{-b}^{b}\left[\int_{-b}^{y}\beta(x)dF(x)\right]d\alpha(y)\,.$$

In addition, one could easily show that $T(y)=-\int_{-b}^{y}\beta(x)dF(x)$ is an even function. Thus, we get

$$\begin{aligned}Cov\big[\alpha(X),\beta(X)\big]&=\int_{-b}^{b}T(y)\,d\alpha(y)\\&=\int_{-b}^{0}T(y)\,d\alpha(y)+\int_{0}^{b}T(y)\,d\alpha(y)\\&=-\int_{0}^{b}T(y)\,d\alpha(-y)+\int_{0}^{b}T(y)\,d\alpha(y)\\&=\int_{0}^{b}T(y)\big[d\big(\alpha(y)-\alpha(-y)\big)\big]\geq 0\,.\end{aligned}$$

The above inequality holds because:

(1) It can easily be shown that $T(y)=-\int_{-b}^{y}\beta(x)dF(x)$ is decreasing and positive for $y\geq 0$, and
(2) $\big(\alpha(y)-\alpha(-y)\big)$ is increasing.

We note that (2) holds because $\alpha(x)$ is an increasing function. Thus, the assertion in Part (a) of Theorem 2.2 holds with $\beta(x)\geq 0$ for all $x\geq 0$. The results for other situations can similarly be proved. □

One may wonder whether the monotonicity assumption for both $\alpha\left(x\right)$ and $\beta(x)$ in Theorem 2.2 could be relaxed. We do this for the Chebyshev inequality as shown in the following theorem:

Theorem 2.3. *Let X be a random variable symmetric about zero with support on $[-b,b]$. Consider two real functions $\alpha(x)$ and $\beta(x)$. Let $\beta(x)$ be an odd function of bounded variation with $\beta(x)\geq(\leq)0$ for all $x\geq 0$. We have*

(1) *if $\alpha(x)\geq\alpha(-x)$ for all $x\geq 0$, then $Cov[\alpha(X),\beta(X)]\geq(\leq)0$; and*
(2) *if $\alpha(x)\leq\alpha(-x)$ for all $x\geq 0$ then $Cov[\alpha(X),\beta(X)]\leq(\geq)0$.*

Proof. We only prove Part (a) of Theorem 2.3 with $\beta(x) \geq 0$ for all $x \geq 0$. We note that since $\beta(x)$ is an odd function and X is a random variable symmetric about zero with support on $[-b, b]$, then $E\big[\beta(X)\big] = 0$. Applying the same steps as shown in the proof of Theorem 2.2, we obtain

$$Cov\big[\alpha(X), \beta(X)\big] = \int_{-b}^{b} \left[- \int_{-b}^{y} \beta(x) dF(x) \right] d\alpha(y) \geq 0 \,.$$

Defining $T(y) = - \int_{-b}^{y} \beta(x) dF(x)$, we have

$$\begin{aligned} Cov\big[\alpha(X), \beta(X)\big] &= \int_{-b}^{b} T(y) \, d\alpha(y) \\ &= - \int_{-b}^{b} \alpha(y) dT(y) + T(b)\alpha(b) - T(-b)\alpha(-b) \,. \end{aligned}$$

As one can easily show that $T(y)$ is an even function, then $T(b) = -E\big[\beta(X)\big] = 0$, $T(-b) = 0$, and we get:

$$\begin{aligned} Cov\big[\alpha(X), \beta(X)\big] &= - \int_{-b}^{b} \alpha(y) dT(y) \\ &= - \int_{-b}^{0} \alpha(y) dT(y) - \int_{0}^{b} \alpha(y) dT(y) \\ &= \int_{0}^{-b} \alpha(y) dT(y) - \int_{0}^{b} \alpha(y) dT(y) \\ &= \int_{0}^{b} \alpha(-y) dT(y) - \int_{0}^{b} \alpha(y) dT(y) \\ &= \int_{0}^{b} \big[\alpha(-y) - \alpha(y)\big] dT(y) \geq 0 \,. \end{aligned}$$

In addition, one can easily show that $T(y)$ is a decreasing function for $y \geq 0$. Moreover, by assumption, $\alpha(-y) - \alpha(y) \leq 0$. Thus, we have $Cov\big[\alpha(X), \beta(X)\big] \geq 0$, and hence, the assertion in Part (a) of Theorem 2.3 follows with $\beta(x) \geq 0$ for all $x \geq 0$. The results for other situations can similarly be proved. □

In the above results, both α and β are functions of the same variable X. We next extend the results such that α and β are functions of two different variables, say X and Y, respectively. However, in order to do this, additional assumptions have to be imposed. In this paper, we assume that both variables have positive quadrant dependency; that is, $H(x, y) - F(x)G(y) \geq 0$.

Theorem 2.4. *Let X and Y be two random variables with positive quadrant dependency. Consider two functions $\alpha(x)$ and $\beta(x)$. We have:*

(1) *if $\alpha(x)$ is increasing (decreasing) and $\beta(x)$ is increasing (decreasing), then*

$$Cov\big[\alpha(X), \beta(Y)\big] \geq 0,$$

(2) *if one of the functions is increasing and the other is decreasing, then*

$$Cov\big[\alpha(X), \beta(Y)\big] \leq 0.$$

Proof. We only prove the second part of Theorem 2.4. The first part of the theorem can be proved similarly. Letting $K(x, y) = H(x, y) - F(x)G(y)$, we have

$$Cov\big[\alpha(X), \beta(Y)\big] = \int_{a}^{b} \int_{a}^{b} K(x, y) \, d\alpha(x) \, d\beta(y) \,.$$

For the situation in which $\alpha(x)$ is an increasing function, since $K(x,y) \geq 0$ is continuous, we have

$$T(y) = \int_a^b K(x,y)\, d\alpha(x) \geq 0\,.$$

In addition, as $\big(-\beta(x)\big)$ is an increasing function, we can easily show that

$$Cov\big[\alpha(X), \beta(Y)\big] = -\int_a^b K(x,y) d\big(-\beta(y)\big) \leq 0\,,$$

and thus the assertion follows. □

We note that reverse results can easily be obtained if one assumes negative quadrant dependency. Therefore, we skip the discussion of properties of the covariance inequality for negative quadrant dependency.

We first developed Theorem 2.2 to relax the monotonicity assumption on the function $\beta(x)$ for Proposition 2.1. We also developed Theorem 2.3 to relax the monotonicity assumption on both $\alpha(x)$ and $\beta(x)$. Thereafter, we developed results for the Chebyshev inequality for two random variables X and Y as shown in Theorem 2.4. We then considered relaxing the monotonicity assumption for Theorem 2.4. To relax the monotonicity assumption on the function(s) for Proposition 2.1, as shown in Theorems 2.2 and 2.3, is easier than for Theorem 2.4 as these theorems deal with only one variable, whereas Theorem 2.4 deals with two random variables X and Y. In this paper, we managed to relax the monotonicity assumption on $\beta(x)$ for Theorem 2.4 as shown in below. We leave the relaxation of the monotonicity assumption on both $\alpha(x)$ and $\beta(x)$ for further study.

Theorem 2.5. *Let X and Y be two dependent random variables with support on $[-b, b]$. Assume $K(x,y) = H(x,y) - F(x)G(y)$ is increasing in y. Consider two functions $\alpha(x)$ and $\beta(x)$, where $\beta(x)$ is an even function of bounded variation increasing (decreasing) for all $x \geq 0$. We have*

(1) *if $\alpha(x)$ is increasing, then $Cov\big[\alpha(X), \beta(Y)\big] \geq (\leq)\ 0$; and*
(2) *if $\alpha(x)$ is decreasing, then $Cov\big[\alpha(X), \beta(Y)\big] \leq (\geq)\ 0$.*

Proof. We only prove the first part. Let

$$Cov\big[\alpha(X), \beta(Y)\big] = \int_{-b}^b \int_{-b}^b K(x,y)\, d\alpha(x)\, d\beta(y)\,.$$

Since $\frac{\partial K}{\partial y} \geq 0$, $K(x,y) - K(x,-y) \geq 0$ for all $y \geq 0$. Using the assumption that $\beta(x)$ is an even function and increasing for $x \geq 0$, we obtain

$$\begin{aligned} T(x) &= \int_{-b}^b K(x,y)\, d\beta(y) \\ &= \int_{-b}^0 K(x,y)\, d\beta(y) + \int_0^b K(x,y)\, d\beta(y) \\ &= -\int_0^{-b} K(x,y)\, d\beta(y) + \int_0^b K(x,y)\, d\beta(y) \\ &= \int_0^b \big[K(x,y) - K(x,-y)\big] d\beta(y) \geq 0 \end{aligned}$$

Finally, as $\alpha(x)$ is an increasing function, we get

$$Cov\big[\alpha(X), \beta(Y)\big] = \int_{-b}^b T(x)\, d\alpha(x) \geq 0\,,$$

and the assertion follows. □

We note that, in this case, we have relaxed the monotonicity assumption of one of the functions.

3. CONCLUSION

We derived some covariance inequalities for monotonic and non-monotonic functions. Although we relaxed the monotonicity assumptions in some of our results, we imposed a symmetry assumption on the random variables and restricted our analysis only to even or odd functions. The analysis of new covariance inequalities without these assumptions remains a task for future research.

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