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ABSOLUTE NÖRLUND SUMMABILITY FACTORS

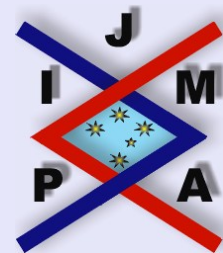
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Abstract

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Abstract

In this paper a theorem on the absolute Nörlund summability factors has been proved under more weaker conditions by using a quasi β -power increasing sequence instead of an almost increasing sequence.

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Key words: Nörlund summability, summability factors, power increasing sequences.

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [2]).

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n -th Cesàro means of order α , with $\alpha > -1$, of the sequences (s_n) and (w_n) , respectively. The series $\sum a_n$ is said to be summable $|C, \alpha|$, if (see [5], [7])

$$(1.1) \quad \sum_{n=1}^{\infty} |u_n^\alpha - u_{n-1}^\alpha| = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha| < \infty.$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$(1.2) \quad P_n = p_0 + p_1 + p_2 + \dots + p_n \neq 0, \quad (n \geq 0).$$

The sequence-to-sequence transformation

$$(1.3) \quad \sigma_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v$$

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be summable $|N, p_n|$, if (see [9])

$$(1.4) \quad \sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty.$$



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In the special case when

$$(1.5) \quad p_n = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)\Gamma(n + 1)}, \quad \alpha \geq 0$$

the Nörlund mean reduces to the (C, α) mean and $|N, p_n|$ summability becomes $|C, \alpha|$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C, 1)$ mean and then $|N, p_n|$ summability becomes $|C, 1|$ summability. For any sequence (λ_n) , we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1}$.

In [6] Kishore has proved the following theorem concerning $|C, 1|$ and $|N, p_n|$ summability methods.

Theorem 1.1. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C, 1|$, then the series $\sum a_n P_n(n+1)^{-1}$ is summable $|N, p_n|$.*

Ahmad [1] proved the following theorem for absolute Nörlund summability factors.

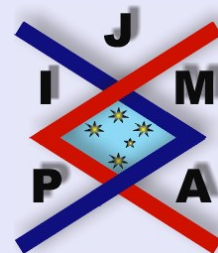
Theorem 1.2. *Let (p_n) be as in Theorem 1.1. If*

$$(1.6) \quad \sum_{v=1}^n \frac{1}{v} |t_v| = O(X_n) \quad \text{as } n \rightarrow \infty,$$

where (X_n) is a positive non-decreasing sequence and (λ_n) is a sequence such that

$$(1.7) \quad X_n \lambda_n = O(1),$$

$$(1.8) \quad n\Delta X_n = O(X_n),$$



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$$(1.9) \quad \sum nX_n |\Delta^2 \lambda_n| < \infty,$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.

Later on Bor [3] proved Theorem 1.2 under weaker conditions in the following form.

Theorem 1.3. Let (p_n) be as in Theorem 1.1 and let (X_n) be a positive non-decreasing sequence. If the conditions (1.6) and (1.7) of Theorem 1.2 are satisfied and the sequences (λ_n) and (β_n) are such that

$$(1.10) \quad |\Delta \lambda_n| \leq \beta_n,$$

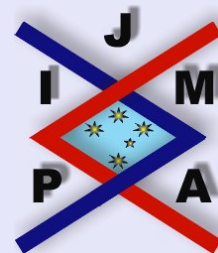
$$(1.11) \quad \beta_n \rightarrow 0,$$

$$(1.12) \quad \sum nX_n |\Delta \beta_n| < \infty,$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.

Also Bor [4] has proved Theorem 1.3 under the weaker conditions in the following form.

Theorem 1.4. Let (p_n) be as in Theorem 1.1 and let (X_n) be an almost increasing sequence. If the conditions (1.6), (1.7), (1.10) and (1.12) of Theorem 1.2 and Theorem 1.3 are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.



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2. Main Result

The aim of this paper is to prove Theorem 1.4 under more weaker conditions. For this we need the concept of a quasi β -power increasing sequence. A positive sequence (γ_n) is said to be a quasi β -power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$(2.1) \quad Kn^\beta \gamma_n \geq m^\beta \gamma_m$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β -power increasing sequence for any nonnegative β , but the converse need not be true as can be seen by taking an example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem replacing an almost increasing sequence by a quasi β -power increasing sequence.

Now we shall prove the following theorem.

Theorem 2.1. *Let (p_n) be as in Theorem 1.1 and let (X_n) be a quasi β -power increasing sequence. If the conditions (1.6), (1.7), (1.10) and (1.12) of Theorem 1.2 and Theorem 1.3 are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.*

We need the following lemma for the proof of our theorem.

Lemma 2.2 ([8]). *Under the conditions on (X_n) , (λ_n) and (β_n) , as taken in the statement of the theorem, the following conditions hold, when (1.12) is satisfied:*

$$(2.2) \quad n\beta_n X_n = O(1) \quad \text{as } n \rightarrow \infty,$$



$$(2.3) \quad \sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Proof of Theorem 2.1. In order to prove the theorem, we need consider only the special case in which (N, p_n) is $(C, 1)$, that is, we shall prove that $\sum a_n \lambda_n$ is summable $|C, 1|$. Our theorem will then follow by means of Theorem 1.1. Let T_n be the n -th $(C, 1)$ mean of the sequence $(na_n \lambda_n)$, that is,

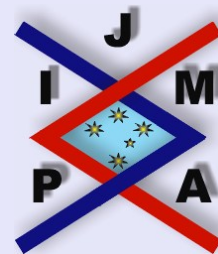
$$(2.4) \quad T_n = \frac{1}{n+1} \sum_{v=1}^n va_v \lambda_v.$$

Using Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1} \sum_{v=1}^n va_v \lambda_v \\ &= \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta \lambda_v (v+1)t_v + \lambda_n t_n \\ &= T_{n,1} + T_{n,2}, \quad \text{say.} \end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$(2.5) \quad \sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}| < \infty \quad \text{for } r = 1, 2, \text{ by (1.1).}$$



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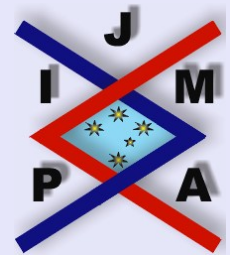
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Now, we have

$$\begin{aligned}
 \sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}| &\leq \sum_{n=2}^{m+1} \frac{1}{n(n+1)} \left\{ \sum_{v=1}^{n-1} \frac{v+1}{v} v |\Delta\lambda_v| |t_v| \right\} \\
 &= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v\beta_v |t_v| \right\} \\
 &= O(1) \sum_{v=1}^m v\beta_v |t_v| \sum_{n=v+1}^{m+1} \frac{1}{n^2} \\
 &= O(1) \sum_{v=1}^m v\beta_v \frac{|t_v|}{v} \\
 &= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^v \frac{|t_r|}{r} + O(1)m\beta_m \sum_{v=1}^m \frac{|t_v|}{v} \\
 &= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1)m\beta_m X_m \\
 &= O(1) \sum_{v=1}^{m-1} |(v+1)\Delta\beta_v - \beta_v| X_v + O(1)m\beta_m X_m \\
 &= O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_v| X_v + O(1)m\beta_m X_m \\
 &= O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned}$$

by (1.6), (1.10), (1.12), (2.2) and (2.3).



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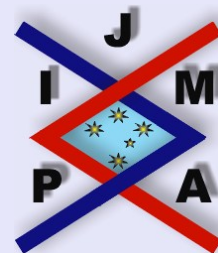
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Again

$$\begin{aligned}
 \sum_{n=1}^m \frac{1}{n} |T_{n,2}| &= \sum_{n=1}^m |\lambda_n| \frac{|t_n|}{n} \\
 &= \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n \frac{|t_v|}{v} + |\lambda_m| \sum_{n=1}^m \frac{|t_n|}{n} \\
 &= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\
 &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned}$$

by (1.6), (1.7), (1.10) and (2.3). This completes the proof of the theorem. \square



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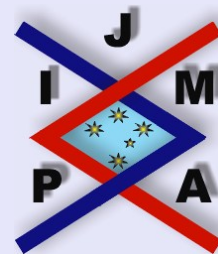
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