Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 5, Article 192, 2006

A SHARP INEQUALITY OF OSTROWSKI-GRÜSS TYPE

ZHENG LIU

INSTITUTE OF APPLIED MATHEMATICS
FACULTY OF SCIENCE
ANSHAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
ANSHAN 114044, LIAONING
PEOPLE'S REPUBLIC OF CHINA.
lewzheng@163.net

Received 9 February, 2006; accepted 23 May, 2006
Communicated by J. Sándor

ABSTRACT. The main purpose of this paper is to use a Grüss type inequality for Riemann-Stieltjes integrals to obtain a sharp integral inequality of Ostrowski-Grüss type for functions whose first derivative are functions of Lipschitizian type and precisely characterize the functions for which equality holds.

Key words and phrases: Ostrowski-Grüss type inequality, Grüss type inequality for Riemann-Stieltjes integrals, Lipschitzian type function, Sharp bound.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

In 1935, G. Grüss (see [4, p. 296]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

Theorem A. *Let* $h, g : [a, b] \to \mathbb{R}$ *be two integrable functions such that* $\phi \leq h(x) \leq \Phi$ *and* $\gamma \leq g(x) \leq \Gamma$ *for all* $x \in [a, b]$, *where* $\phi, \Phi, \gamma, \Gamma$ *are real numbers. Then we have*

$$(1.1)\qquad |T(h,g)| := \left|\frac{1}{b-a}\int_a^b h(x)g(x)dx - \frac{1}{b-a}\int_a^b h(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ *cannot be replaced by a smaller one.*

ISSN (electronic): 1443-5756

163-06

It is clear that the constant $\frac{1}{4}$ is achieved for

$$h(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2}\right).$$

From then on, (1.1) has been known in the literature as the Grüss inequality.

In 1998, S.S. Dragomir and I. Fedotov [2] established the following Grüss type inequality for Riemann-Stieltjes integrals:

Theorem B. *Let $h, u : [a,b] \to \mathbb{R}$ be so that u is L-Lipschitzian on $[a,b]$, i.e.,*

$$|u(x) - u(y)| \leq L\,|x - y|$$

for all $x, y \in [a,b]$, h is Riemann integrable on $[a,b]$ and there exists the real numbers m, M so that $m \leq h(x) \leq M$ for all $x \in [a,b]$. Then we have the inequality

$$\left|\int_a^b h(x)du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b h(t)dt\right| \leq \frac{1}{2}L(M-m)(b-a) \tag{1.2}$$

and the constant $\frac{1}{2}$ is sharp.

In a recent paper [3], the inequality (1.2) has been improved and refined as follows:

Theorem C. *Let $h, u : [a,b] \to \mathbb{R}$ be so that u is L-Lipschitzian on $[a,b]$, h is Riemann integrable on $[a,b]$ and there exist the real numbers m, M so that $m \leq h(x) \leq M$ for all $x \in [a,b]$. Then we have*

$$\begin{aligned}\left|\int_a^b h(x)du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b h(t)dt\right| &\leq L\int_a^b \left|h(x) - \frac{1}{b-a}\int_a^b h(t)dt\right|dx \\ &\leq L(b-a)\sqrt{T(h,h)} \\ &\leq \frac{1}{2}L(M-m)(b-a).\end{aligned} \tag{1.3}$$

All the inequalities in (1.3) are sharp and the constant $\frac{1}{2}$ is the best possible one.

Theorem D. *Let $h, u : [a,b] \to \mathbb{R}$ be so that u is (l, L)-Lipschitzian on $[a,b]$, i.e., it satisfies the condition*

$$l(x_2 - x_1) \leq u(x_2) - u(x_1) \leq L(x_2 - x_1)$$

for $a \leq x_1 \leq x_2 \leq b$ with $l < L$, h is Riemann integral on $[a,b]$ and there exist the real numbers m, M so that $m \leq h(x) \leq M$ for all $x \in [a,b]$. Then we have the inequality

$$\begin{aligned}\left|\int_a^b h(x)du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b h(t)dt\right| &\leq \frac{L-l}{2}\int_a^b \left|h(x) - \frac{1}{b-a}\int_a^b h(t)dt\right|dx \\ &\leq \frac{L-l}{2}(b-a)\sqrt{T(h,h)} \\ &\leq \frac{1}{4}(L-l)(M-m)(b-a).\end{aligned} \tag{1.4}$$

All the inequalities in (1.4) are sharp and the constant $\frac{1}{4}$ is the best possible one.

In [1], L.J. Dedić et al. have proved the following Ostrowski type inequality as

Theorem E. *If u' is L-Lipschitzian on $[a,b]$, then for every $x \in [a,b]$ we have*

$$\left|\int_a^b u(t)dt - \frac{b-a}{2}\left[u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2}\right)u'(x)\right]\right| \leq L\left(\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\right). \tag{1.5}$$

In this paper, we will use Theorem C and Theorem D to obtain some sharp integral inequalities of Ostrowski-Grüss type for functions whose first derivative are functions of Lipschitzian type. Thus a further generalization of the Ostrowski type inequality and a perturbed version of the inequality (1.5) is obtained.

2. THE RESULTS

Theorem 2.1. *Let $u : [a,b] \to \mathbb{R}$ be a differentiable function so that u' is (l,L)-Lipschitzian on $[a,b]$, i.e., satisfies the condition*

$$l(x_2 - x_1) \leq u'(x_2) - u'(x_1) \leq L(x_2 - x_1) \tag{2.1}$$

for $a \leq x_1 \leq x_2 \leq b$ with $l < L$.Then for all $x \in [a,b]$ we have

$$\left|\int_a^b u(t)dt - \frac{b-a}{2}\left[(u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2}\right)u'(x)\right] - \frac{u'(b)-u'(a)}{4}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]\right| \leq \frac{L-l}{4} I(a,b,x), \tag{2.2}$$

where

$$I(a,b,x) = \begin{cases} \frac{1}{6}\left(\frac{a+b}{2} - x\right)\left(\frac{a+3b}{4} - x\right)[3(x-a) + (b-x)] \\ \qquad + \frac{4}{3}\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}, & a \leq x \leq \xi, \\ \frac{1}{6}\left(\frac{a+b}{2} - x\right)\left(x - \frac{3a+b}{4}\right)[(x-a) + 3(b-x)] \\ \qquad + 4\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}, & \xi < x < \zeta, \\ \frac{16}{3}\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}, & \zeta \leq x \leq \theta, \\ \frac{1}{6}\left(x - \frac{a+b}{2}\right)\left(\frac{a+3b}{4} - x\right)[3(x-a) + (b-x)] \\ \qquad + 4\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}, & \theta < x < \eta, \\ \frac{1}{6}\left(x - \frac{a+b}{2}\right)\left(x - \frac{3a+b}{4}\right)[(x-a) + 3(b-x)] \\ \qquad + \frac{4}{3}\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}, & \eta \leq x \leq b \end{cases} \tag{2.3}$$

with

$$\xi = \frac{a+b}{2} - \frac{\sqrt{3}(b-a)}{6}, \qquad \eta = \frac{a+b}{2} + \frac{\sqrt{3}(b-a)}{6},$$
$$\zeta = a + \frac{\sqrt{6}(b-a)}{6}, \qquad \theta = b - \frac{\sqrt{6}(b-a)}{6}$$

and $a < \xi < \frac{3a+b}{4} < \zeta < \frac{a+b}{2} < \theta < \frac{a+3b}{4} < \eta < b$.

Proof. Integrating by parts produces the identity

$$\int_a^b K(x,t)du'(t) = \int_a^b u(t)dt - \frac{1}{2}(b-a)\left[u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2}\right)u'(x)\right], \tag{2.4}$$

where

$$K(x,t) = \begin{cases} \frac{1}{2}(t-a)\left(t - \frac{a+b}{2}\right), & t \in [a,x], \\ \frac{1}{2}(t-b)\left(t - \frac{a+b}{2}\right), & t \in (x,b]. \end{cases} \tag{2.5}$$

Moreover,

$$\frac{1}{b-a}\int_a^b K(x,t)dt = \frac{1}{4}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]. \tag{2.6}$$

Applying the Grüss type inequality (1.4) gives

$$\left|\int_a^b K(x,t)du'(t) - \frac{u'(b)-u'(a)}{b-a}\int_a^b K(x,t)dt\right| \le \frac{L-l}{2}\int_a^b \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right|dt.$$

Then for any fixed $x \in [a,b]$ we can derive from (2.4), (2.5) and (2.6) that

$$\left|\int_a^b u(t)dt - \frac{b-a}{2}\left[u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2}\right)u'(x)\right] - \frac{u'(b)-u'(a)}{4}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]\right| \le \frac{L-l}{4} I(a,b,x), \tag{2.7}$$

where

$$I(a,b,x) = \int_a^x \left|(t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]\right|dt + \int_x^b \left|(t-b)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right]\right|dt.$$

The last two integrals can be calculated as follows:

For brevity, we put

$$p_1(t) := (t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right], \quad t \in [a,x],$$

$$p_2(t) := (t-b)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right], \quad t \in [x,b].$$

Then we have

$$p_1(a) = p_2(b) = \frac{1}{2}\left[\frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right];$$

$$p_1(x) = \frac{1}{2}\left(x + \frac{b-a}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{(b-a)^2}{24},$$

$$p_2(x) = \frac{1}{2}\left(x - \frac{b-a}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{(b-a)^2}{24}.$$

Set

$$\xi = \frac{a+b}{2} - \frac{\sqrt{3}(b-a)}{6}, \qquad \eta = \frac{a+b}{2} + \frac{\sqrt{3}(b-a)}{6},$$

$$\zeta = a + \frac{\sqrt{6}(b-a)}{6}, \qquad \theta = b - \frac{\sqrt{6}(b-a)}{6}.$$

It is easy to find that $p_1(a) = p_2(b) \le 0$ for $x \in [a, \xi] \cup [\eta, b]$, $p_1(a) = p_2(b) > 0$ for $x \in (\xi, \eta)$ and $p_1(x) \le 0$ for $x \in [a, \zeta]$, $p_1(x) > 0$ for $x \in (\zeta, b]$, $p_2(x) > 0$ for $x \in [a, \theta)$, $p_2(x) \le 0$ for $x \in [\theta, b]$. Notice that

$$a < \xi < \frac{3a+b}{4} < \zeta < \frac{a+b}{2} < \theta < \frac{a+3b}{4} < \eta < b,$$

we see that there are five possible cases to be determined.

(i) In case $x \in [\zeta, \theta]$. $p_1(a) = p_2(b) > 0$, $p_1(x) \ge 0$, $p_2(x) \ge 0$ and it is easy to find by elementary calculus that the function $p_1(t)$ is strictly decreasing in $\left(a, \frac{3a+b}{4}\right)$ and strictly increasing in $\left(\frac{3a+b}{4}, x\right)$, also, as the function $p_2(t)$ is strictly decreasing in $\left(x, \frac{a+3b}{4}\right)$ and strictly increasing in $\left(\frac{a+3b}{4}, b\right)$. Moreover, $p_1\left(\frac{3a+b}{4}\right) = p_2\left(\frac{a+3b}{4}\right) < 0$. So, $p_1(t)$ has two zeros in (a, x) at the points

$$t_1 = \frac{3a+b}{4} - \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{1}{2}}$$

and

$$t_2 = \frac{3a+b}{4} + \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{1}{2}}.$$

Also $p_2(t)$ has two zeros in (x, b) at the points

$$t_3 = \frac{a+3b}{4} - \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{1}{2}}$$

and

$$t_4 = \frac{a+3b}{4} + \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{1}{2}}.$$

Thus we have

$$\begin{aligned} I(a,b,x) = &\int_a^{t_1}\left[(t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{24}\right]dt \\ &+ \int_{t_1}^{t_2}\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24} - (t-a)\left(t - \frac{a+b}{2}\right)\right]dt \\ &+ \int_{t_2}^{x}\left[(t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{24}\right]dt \end{aligned} \tag{2.8}$$

$$+\int_x^{t_3}\left[(t-b)\left(t-\frac{a+b}{2}\right)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]dt$$
$$+\int_{t_3}^{t_4}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{24}-(t-b)\left(t-\frac{a+b}{2}\right)\right]dt$$
$$+\int_{t_4}^{b}\left[(t-b)\left(t-\frac{a+b}{2}\right)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]dt$$
$$=\frac{16}{3}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{48}\right]^{\frac{3}{2}}.$$

(ii) In case $x\in[a,\xi]$, $\quad p_1(a)=p_2(b)\le 0$, $p_1(x)<0$,$p_2(x)>0$ and $p_1(t)$ is strictly decreasing in (a,x) as well as $p_2(t)$ is strictly decreasing in $(x,\frac{a+3b}{4})$ and strictly increasing in $(\frac{a+3b}{4},b)$ with $t_3\in(x,\frac{a+3b}{4})$ such that $p_2(t_3)=0$. Thus we have

$$I(a,b,x)=\int_a^{x}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{24}-(t-a)\left(t-\frac{a+b}{2}\right)\right]dt \tag{2.9}$$
$$+\int_x^{t_3}\left[(t-b)\left(t-\frac{a+b}{2}\right)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]dt$$
$$+\int_{t_3}^{b}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{24}-(t-b)\left(t-\frac{a+b}{2}\right)\right]dt$$
$$=\frac{1}{6}\left(\frac{a+b}{2}-x\right)\left(\frac{a+3b}{4}-x\right)[3(x-a)+(b-x)]$$
$$+\frac{4}{3}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{48}\right]^{\frac{3}{2}}.$$

(iii) In case (ξ,ζ), $p_1(a)=p_2(b)>0$, $p_1(x)<0$,$p_2(x)>0$ and $p_1(t)$ has a unique zero $t_1\in(a,x)$, $p_2(t)$ has two zeros $t_3,t_4\in(x,b)$. Thus we have

$$I(a,b,x)=\int_a^{t_1}\left[(t-a)\left(t-\frac{a+b}{2}\right)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]dt \tag{2.10}$$
$$+\int_{t_1}^{x}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{24}-(t-a)\left(t-\frac{a+b}{2}\right)\right]dt$$
$$+\int_x^{t_3}\left[(t-b)\left(t-\frac{a+b}{2}\right)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]dt$$
$$+\int_{t_3}^{t_4}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2-\frac{(b-a)^2}{24}-(t-b)\left(t-\frac{a+b}{2}\right)\right]dt$$
$$+\int_{t_4}^{b}\left[(t-b)\left(t-\frac{a+b}{2}\right)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{24}\right]dt$$

$$= \frac{1}{6}\left(\frac{a+b}{2} - x\right)\left(x - \frac{3a+b}{4}\right)[(x-a) + 3(b-x)] + 4\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}.$$

(iv) In case $x \in (\theta, \eta)$, $p_1(a) = p_2(b) > 0, p_1(x) > 0, p_2(x) < 0$ and $p_1(t)$ has two zeros $t_1, t_2 \in (a, x)$, $p_2(t)$ has a unique zero $t_4 \in (x, b)$. Thus we have

$$\begin{aligned} I(a,b,x) &= \int_a^{t_1} \left[(t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{24}\right] dt \\ &\quad + \int_{t_1}^{t_2} \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24} - (t-a)\left(t - \frac{a+b}{2}\right)\right] dt \\ &\quad + \int_{t_2}^{x} \left[(t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{24}\right] dt \\ &\quad + \int_{x}^{t_4} \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24} - (t-b)\left(t - \frac{a+b}{2}\right)\right] dt \\ &\quad + \int_{t_4}^{b} \left[(t-b)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{24}\right] dt \\ &= \frac{1}{6}\left(x - \frac{a+b}{2}\right)\left(\frac{a+3b}{4} - x\right)[3(x-a) + (b-x)] \\ &\quad + 4\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}. \end{aligned} \tag{2.11}$$

(v) In case $x \in [\eta, b]$, $p_1(a) = p_2(b) \le 0, p_1(x) > 0, p_2(x) < 0$ and $p_1(t)$ has a unique zero $t_2 \in (a, x)$, $p_2(t) \le 0$ for $t \in [x, b]$. Thus we have

$$\begin{aligned} I(a,b,x) &= \int_a^{t_2} \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24} - (t-a)\left(t - \frac{a+b}{2}\right)\right] dt \\ &\quad + \int_{t_2}^{x} \left[(t-a)\left(t - \frac{a+b}{2}\right) - \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{24}\right] dt \\ &\quad + \int_{x}^{b} \left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24} - (t-b)\left(t - \frac{a+b}{2}\right)\right] dt \\ &= \frac{1}{6}\left(x - \frac{a+b}{2}\right)\left(x - \frac{3a+b}{4}\right)[(x-a) + 3(b-x)] \\ &\quad + \frac{4}{3}\left[\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{48}\right]^{\frac{3}{2}}. \end{aligned} \tag{2.12}$$

Consequently, the inequality (2.2) with (2.3) follows from (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12).

The proof is completed. □

Remark 2.2. It is not difficult to prove that the inequality (2.2) with (2.3) is sharp in the sense that we can construct the function u to attain the equality in (2.2) with (2.3). Indeed, we may choose u such that

$$u(t) = \begin{cases} \frac{1}{2}(t-a)^2, & a \le t < x, \\ \frac{L}{2}(t-x)^2 + \frac{l}{2}[2(x-a)t - (x^2-a^2)], & x \le t < t_3, \\ \frac{l}{2}[(t-t_3)^2 + 2(x-a)t - (x^2-a^2)] & \\ \qquad + \frac{L}{2}[2(t_3-x)t - (t_3^2 - x^2)], & t_3 \le t \le b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} l(t-a), & a \le t < x, \\ L(t-x) + (x-a)l, & x \le t < t_3, \\ l(t - t_3 + x - a) + (t_3 - x)L, & t_3 \le t \le b, \end{cases}$$

for any $x \in [a, \xi]$, and

$$u(t) = \begin{cases} \frac{L}{2}(t-a)^2, & a \le t < t_1, \\ \frac{l}{2}(t-t_1)^2 + \frac{L}{2}[2(t_1-a)t - (t_1^2 - a^2)], & t_1 \le t < x, \\ \frac{L}{2}[(t-x)^2 + 2(t_1-a)t - (t_1^2-a^2)] & \\ \qquad + \frac{l}{2}[2(x-t_1)t - (x^2 - t_1^2)], & x \le t < t_3, \\ \frac{l}{2}[(t-t_3^2) + 2(x-t_1)t - (x^2-t_1^2)] & \\ \qquad + \frac{L}{2}[2(t_3 - x + t_1 - a)t - (t_3^2 - x^2 + t_1^2 - a^2)], & t_3 \le t < t_4, \\ \frac{L}{2}[(t-t_4)^2 + 2(t_3 - x + t_1 - a)t - (t_3^2 - x^2 + t_1^2 - a^2)] & \\ \qquad + \frac{l}{2}[2(t_4 - t_3 + x - t_1)t - (t_4^2 - t_3^2 + x^2 - t_1^2)], & t_4 \le t \le b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} L(t-a), & a \le t < t_1, \\ l(t-t_1) + (t_1-a)L, & t_1 \le t < x, \\ L(t - x + t_1 - a) + (x - t_1)l, & x \le t < t_3, \\ l(t - t_3 + x - t_1) + (t_3 - x + t_1 - a)L, & t_3 \le t < t_4, \\ L(t - t_4 + t_3 - x + t_1 - a) + (t_4 - t_3 + x - t_1)l, & t_4 \le t \le b, \end{cases}$$

for any $x \in (\xi, \zeta)$, and

$$u(t) = \begin{cases} \frac{L}{2}(t-a)^2, & a \le t < t_1, \\ \frac{l}{2}(t-t_1)^2 + \frac{L}{2}[2(t_1-a)t - (t_1^2 - a^2)], & t_1 \le t < t_2, \\ \frac{L}{2}[(t-t_2)^2 + 2(t_1-a)t - (t_1^2-a^2)] \\ \qquad + \frac{l}{2}[2(t_2-t_1)t - (t_2^2 - t_1^2)], & t_2 \le t < t_3, \\ \frac{l}{2}[(t-t_3^2) + 2(t_2-t_1)t - (t_2^2-t_1^2)] \\ \qquad + \frac{L}{2}[2(t_3-t_2+t_1-a)t - (t_3^2 - t_2^2 + t_1^2 - a^2)], & t_3 \le t < t_4, \\ \frac{L}{2}[(t-t_4)^2 + 2(t_3-t_2+t_1-a)t - (t_3^2-t_2^2+t_1^2-a^2)] \\ \qquad + \frac{l}{2}[2(t_4-t_3+t_2-t_1)t - (t_4^2-t_3^2+t_2^2-t_1^2)], & t_4 \le t \le b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} L(t-a), & a \le t < t_1, \\ l(t-t_1) + (t_1-a)L, & t_1 \le t < t_2, \\ L(t-t_2+t_1-a) + (t_2-t_1)l, & t_2 \le t < t_3, \\ l(t-t_3+t_2-t_1) + (t_3-t_2+t_1-a)L, & t_3 \le t < t_4, \\ L(t-t_4+t_3-t_2+t_1-a) + (t_4-t_3+t_2-t_1)l, & t_4 \le t \le b, \end{cases}$$

for any $x \in (\xi, \zeta)$, and

$$u(t) = \begin{cases} \frac{L}{2}(t-a)^2, & a \le t < t_1, \\ \frac{l}{2}(t-t_1)^2 + \frac{L}{2}[2(t_1-a)t - (t_1^2 - a^2)], & t_1 \le t < t_2, \\ \frac{L}{2}[(t-t_2)^2 + 2(t_1-a)t - (t_1^2-a^2)] \\ \qquad + \frac{l}{2}[2(t_2-t_1)t - (t_2^2 - t_1^2)], & t_2 \le t < x, \\ \frac{l}{2}[(t-x^2) + 2(t_2-t_1)t - (t_2^2-t_1^2)] \\ \qquad + \frac{L}{2}[2(x-t_2+t_1-a)t - (x^2 - t_2^2 + t_1^2 - a^2)], & x \le t < t_4, \\ \frac{L}{2}[(t-t_4)^2 + 2(x-t_2+t_1-a)t - (x^2-t_2^2+t_1^2-a^2)] \\ \qquad + \frac{l}{2}[2(t_4-x+t_2-t_1)t - (t_4^2-x^2+t_2^2-t_1^2)], & t_4 \le t \le b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} L(t-a), & a \le t < t_1, \\ l(t-t_1) + (t_1-a)L, & t_1 \le t < t_2, \\ L(t-t_2+t_1-a) + (t_2-t_1)l, & t_2 \le t < x, \\ l(t-x+t_2-t_1) + (x-t_2+t_1-a)L, & x \le t < t_4, \\ L(t-t_4+x-t_2+t_1-a) + (t_4-x+t_2-t_1)l, & t_4 \le t \le b, \end{cases}$$

for any $x \in (\theta, \eta)$, and

$$u(t) = \begin{cases} \frac{l}{2}(t-a)^2, & a \le t < t_2, \\ \frac{L}{2}(t-t_2)^2 + \frac{l}{2}[2(t_2-a)t - (t_2^2 - a^2)], & t_2 \le t < x, \\ \frac{l}{2}[(t-x)^2 + 2(t_2-a)t - (t_2^2 - a^2)] & \\ \qquad + \frac{L}{2}[2(x-t_2)t - (x^2 - t_2^2)], & x \le t \le b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} l(t-a), & a \le t < t_2, \\ L(t-t_2) + (t_2-a)l, & t_2 \le t < x, \\ l(t-x+t_2-a) + (x-t_2)L, & x \le t \le b. \end{cases}$$

for any $x \in [\eta, b]$.

It is clear that all the above $u'(t)$ satisfy the condition (2.1) on $[a, b]$.

Remark 2.3. For $x = \frac{a+b}{2}$, we have

$$\left| \int_a^b u(t)dt - \frac{b-a}{2} \left[u\left(\frac{a+b}{2}\right) + \frac{u(a)+u(b)}{2} \right] + \frac{(b-a)^2}{48}[u'(b) - u'a)] \right| \le \frac{(L-l)(b-a)^3}{144\sqrt{3}}.$$

Corollary 2.4. *If u' is L-Lipschitzian on $[a, b]$, then for all $x \in [a, b]$ we have*

$$\left| \int_a^b u(t)dt - \frac{b-a}{2} \left[u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2}\right) u'(x) \right] - \frac{u'(b)-u'(a)}{4} \left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12} \right] \right| \le \frac{L}{2} I(a, b, x), \tag{2.13}$$

where $I(a, b, x)$ is as defined in (2.3).

Proof. It is immediate by taking $l = -L$ in the theorem. □

REFERENCES

[1] L.J. DEDIĆ, M. MATIĆ, J. PEČARIĆ AND A. VUKELIĆ, On generalizations of Ostrowski inequality via Euler harmonic identities, *J. of Inequal. & Appl.*, **7**(6) (2002), 787–805.

[2] S.S. DRAGOMIR AND I. FEDOTOV, An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4) (1998), 286–292.

[3] Z. LIU, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. of Math.*, **30**(4) (2004), 483–489.

[4] D.S. MITRINOVIĆ, J. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.