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PROOF FOR A CONJECTURE ON GENERAL MEANS

PHAN THANH NAM AND MACH NGUYET MINH

DEPARTMENT OF MATHEMATICS
HOCHIMINH CITY NATIONAL UNIVERSITY
227 NGUYEN VAN CU, Q5
HOCHIMINH CITY, VIETNAM
ptnam373@yahoo.com

minhufo@yahoo.com

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ABSTRACT. We give a proof for a conjecture suggested by Olivier de La Grandville and Robert M. Solow, which says that the general mean of two positive numbers, as a function of its order, has one and only one inflection point.

Key words and phrases: General means, Inflection point.

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1. INTRODUCTION

Let $x_1, x_2, ..., x_n$ and $f_1, f_2, ..., f_n$ be $2n$ positive numbers, $\sum_{i=1}^n f_j = 1$. We define the general mean (power mean) function

$$M(p) = \left(\sum_{i=1}^n f_i x_i^p\right)^{\frac{1}{p}}, \quad -\infty < p < \infty.$$

It is well-known that $M(p)$ is smooth, increasing and

$$\min\{x_1, ..., x_n\} = M(-\infty) \le M(p) \le M(+\infty) = \max\{x_1, ..., x_n\}$$

(see, e.g, [4]). However, the exact shape of the curve $M(p)$ in (M, p) space, which relates to the second derivative, has not yet been uncovered.

Note that

$$pM'(p) = M(p)\left(-\ln(M(p)) + \frac{\sum_{i=1}^n f_i \ln(x_i) x_i^p}{\sum_{i=1}^n f_i x_i^p}\right)$$

is bounded, and hence $M'(-\infty) = M'(+\infty) = 0$. Consequently, $M(p)$ has at least one inflection point.

Recently, Olivier de La Grandville and Robert M. Solow [1] conjectured that if $n = 2$ then $M(p)$ has one and only one inflection point; moreover, between its limiting values, $M(p)$ is in a

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first phase convex and then turns concave. These authors also explained the importance of this conjecture in today's economies [1, 2], but they could not offer an analytical proof due to the extreme complexity of the second derivative.

Our aim is to give a proof for this conjecture. Rigorously, we shall prove the following result.

Theorem 1.1. *Assume that $n = 2$ and $x_1 \neq x_2$. Then there exists a unique point $p_0 \in (-\infty, +\infty)$ such that $M''(p_0) = 0$, $M''(p) > 0$ if $p < p_0$, and $M''(p) < 0$ if $p > p_0$.*

Because $M'(p) > 0$ and $M'(-\infty) = M'(+\infty) = 0$, it is sufficient to prove that $M''(p) = 0$ has at most one solution. Note that this result cannot be extended to $n > 2$ (see [1, 3]).

2. PROOF

The proof is divided into three steps. First, we make a change of variables and transform the original problem into a problem of proving the positivity of a two-variable function. Second, we use a simple scheme to reduce this problem to the one of verifying the positivity of some one-variable functions. Finally, we use the same scheme to accomplish the rest.

Step 1. We can assume that $x_2 > x_1$ and write

$$M(p) = x_1(1 - t + tx^p)^{\frac{1}{p}},$$

where $x = x_2/x_1 > 1$, $t = f_2 \in (0, 1)$. Put

$$U(p) = \ln(M(p)) = \frac{\ln(1 - t + tx^p)}{p} + \ln(x_1).$$

Note that $M'(p) = M(p)U'(p) > 0$ and

$$M''(p) = (\exp(U))'' = \exp(U)\,(U')^2 \left(\frac{U''}{(U')^2} + 1 \right).$$

In order to prove that $M''(p) = 0$ has at most one solution, we shall show that the function $U''/\,(U')^2$ is strictly decreasing. It is equivalent to

$$2\,(U'')^2 - U'U''' > 0, \qquad p \in (-\infty, 0) \cup (0, +\infty).$$

It suffices to prove the latter equality only for $p > 0$ because of the symmetry

$$[U'(p)]_{t=r} = [U'(-p)]_{t=1-r}\,, \qquad 0 < r < 1.$$

Making a change of variables, $h = x^p$, $k = 1 - t + th$, we shall prove that the function

$$\begin{aligned}
u(h,k) &= k^3(h-1)^3 \left[p^6 \left(2\,(U'')^2 - U'U''' \right) \right]_{p=\frac{\ln(h)}{\ln(x)},\, t=\frac{k-1}{h-1}} \\
&= h^2(k-1)^2(h-k)\,(\ln(h))^4 \\
&\quad - h(k-1)(h-k)((hk + k - 2h)\ln(k) + 5hk - 5h)\,(\ln(h))^3 \\
&\quad - kh(h-1)(k-1)(-2hk + 5k\ln(k) + 2h - 5h\ln(k))\,(\ln(h))^2 \\
&\quad - 4k^2 h\ln(k)(h-1)^2(k-1)\ln(h) + 2\,(\ln(k))^2\, k^3(h-1),
\end{aligned}$$

is positive. Here $h > k > 1$ since $p > 0$ and $x > 1 > t > 0$.

It remains to show that $u(h, k) > 0$ when $h > k > 1$. This function is a polynomial in terms of h, k, $\ln(h)$ and $\ln(k)$, and the appearance of the logarithm functions make it intractable.

Step 2. To tackle the problem, we need to reduce gradually the order of the logarithm functions. Our main tool is a simple scheme given by the following lemma.

Lemma 2.1. *Let a be a real constant, $m \geq 1$ be an integer, and let $v(s)$, $g_i(s)$, $i = 0, 1, ..., m-1$, be m^{th}-differentiable functions in $[a, +\infty)$. Define a sequence $\{v_i(s)\}_{i=0}^{m}$ by*

$$v_0(s) = v(s), \quad v_{i+1}(s) = (g_i v_i)'(s), \; i = 0, 1, ..., m-1.$$

Assume, for all $s > a$ and $i = 0, 1, ..., m-1$, that

$$g_i(s) > 0, \; v_i(a) \geqslant 0, \quad \textit{and} \quad v_m(s) > 0.$$

Then $v(s) > 0$ for all $s > a$.

Proof. The function $s \mapsto g_{m-1}(s)v_{m-1}(s)$ is strictly increasing because

$$(g_{m-1}v_{m-1})'(s) = v_m(s) > 0, \quad s > a.$$

Therefore $g_{m-1}(s)v_{m-1}(s) > g_{m-1}(a)v_{m-1}(a) \geq 0$ and, consequently, $v_{m-1}(s) > 0$ for all $s > a$. By induction we obtain that $v(s) = v_0(s) > 0$ for all $s > a$. □

We now return to the problem of verifying that $u(h,k) > 0$ when $h > k > 1$. We shall fix $k > 1$ and consider $u(h,k)$ as a one-variable function in terms of h. Choosing $v(h) = u(h,k)$, $a = k$, $m = 13$, $g_i(h) = h^3$ for $i = 4, 7, 10$, and $g_i(h) = 1$ for other cases, we take the sequence $\{v_i\}_{i=0}^{13}$ as in Lemma 2.1. Although the computations seem heavy, they are straightforward and can be implemented easily by mathematics software such as Maple. We find that

$$\begin{aligned}
v_0(k) &= u(k,k) = 0,\\
v_1(k) &= \frac{\partial u}{\partial h}(k,k) = 0,\\
v_2(k) &= \frac{\partial^2 u}{\partial h^2}(k,k)\\
&= 2k(k-1)\left[(\ln(k))^4 + (4+k)(\ln(k))^3 + (7-5k)(\ln(k))^2\right]\\
&\qquad + 2k(k-1)\left[(4-4k)\ln(k) + 2(k-1)^2\right],\\
v_3(k) &= \frac{\partial^3 u}{\partial h^3}(k,k)\\
&= 6(k-1)(\ln(k))^4 + (9k^2+30k-39)(\ln(k))^3 + (126k-27k^2-87)(\ln(k))^2\\
&\qquad + (168k-72-96k^2)\ln(k) + 12(2k-1)(k-1)^2,\\
v_4(k) &= \frac{\partial^4 u}{\partial h^4}(k,k)\\
&= \frac{4(k-1)}{k}\left[(2k+7)(\ln(k))^3 + (8k+40)(\ln(k))^2\right]\\
&\qquad + \frac{4(k-1)}{k}\left[(68-50k)\ln(k) + 11k^2 - 40k + 29\right],\\
v_5(k) &= \frac{\partial}{\partial h}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)(k,k)\\
&= 2k(k-1)\left[(7k+17)(\ln(k))^3 + (58k+130)(\ln(k))^2\right]\\
&\qquad + 2k(k-1)\left[(348-140k)\ln(k) + 56k^2 - 320k + 264\right],
\end{aligned}$$

$$\begin{aligned}
v_6(k) &= \frac{\partial^2}{\partial h^2}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)(k,k)\\
&= 4(k-1)\left[(3k+6)\left(\ln(k)\right)^3+(45+48k)\left(\ln(k)\right)^2\right]\\
&\qquad\qquad +4(k-1)\left[(21k+162)\ln(k)+40k^2-263k+223\right],\\
v_7(k) &= \frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)(k,k)\\
&= \frac{2(k-1)}{k}\left[(30k+6)\left(\ln(k)\right)^2+(251k-101)\ln(k)+(24k+71)(k-1)\right],\\
v_8(k) &= \frac{\partial}{\partial h}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)\right)(k,k)\\
&= 2k(k-1)\left[6(11k-1)\left(\ln(k)\right)^2+(581k-211)\ln(k)+(48k+401)(k-1)\right],\\
v_9(k) &= \frac{\partial^2}{\partial h^2}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)\right)(k,k)\\
&= 12(k-1)\left[12k\left(\ln(k)\right)^2+(131k-57)\ln(k)+(8k+177)(k-1)\right],\\
v_{10}(k) &= \frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)\right)(k,k)\\
&= \frac{12(k-1)}{k}\left[(39k-21)\ln(k)+169(k-1)\right],\\
v_{11}(k) &= \frac{\partial}{\partial h}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)\right)\right)(k,k)\\
&= 12k(k-1)\left[(75k-49)\ln(k)+393(k-1)\right],\\
v_{12}(k) &= \frac{\partial^2}{\partial h^2}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)\right)\right)(k,k)\\
&= 144(k-1)\left[(6k-4)\ln(k)+41(k-1)\right],\\
v_{13}(h) &= \frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^3}{\partial h^3}\left(h^3\frac{\partial^4 u}{\partial h^4}\right)\right)\right)(h,k)\\
&= \frac{48(k-1)^2(24h-k)}{h^2}.
\end{aligned}$$

It is clear that $v_{13}(h)>0$ for $h>k$, and $v_i(k)\ge 0$ for $i=0,1,7,8,...,12$. Therefore, to deduce from Lemma 2.1 that $u(h,k)=v(h)>0$, it remains to check that $v_i(k)\ge 0$ for $i=2,3,4,5,6$.

Step 3. To accomplish the task, we prove that $v_j(k)\ge 0$ for $k>1$, $j=2,3,4,5,6$. For each j, we shall use Lemma 2.1 again with $s=k$, $a=1$, $v=y_j$ which derives from v_j, and $\{g_i\}=\{g_{ji}\}$ chosen appropriately.

For $j=2$, choose

$$y_2(k)=\frac{v_2(k)}{2k(k-1)},\qquad \{g_{2i}(k)\}_{i=0}^5=\{1,1,1,k^2,1,k^2\}.$$

Then $y_2(1)=y_2'(1)=y_2''(1)=(k^2y_2'')'(1)=(k^2y_2'')''(1)=0$ and

$$\left(k^2(k^2y_2'')''\right)'=\frac{4}{k}\left[(3k+12)\ln(k)+(2k+3)(k-1)\right]>0,\qquad k>1.$$

It follows from Lemma 2.1 that $y_2(k)>0$ and, consequently, $v_2(k)>0$ for $k>1$.

For $j = 3$, choose

$$y_3(k) = v_3(k),\ \{g_{3i}(k)\}_{i=0}^{6} = \{1,1,1,k^2,1,1,k^3\}.$$

Then $y_3(1) = y_3'(1) = y_3''(1) = (k^2y_3'')'(1) = (k^2y_3'')''(1) = (k^2y_3'')'''(1) = 0$ and

$$\left(k^3(k^2y_3'')'''\right)'(k) = \frac{12}{k}\left[6k(3k-1)(\ln(k))^2 + (90k^2 - 39k + 24)\ln(k)\right] + \frac{12}{k}\left(216k^3 - 19k^2 - 51k - 21\right) > 0,\ k > 1.$$

It follows from Lemma 2.1 that $v_3(k) = y_3(k) > 0$ for $k > 1$.

For $j = 4$, choose

$$y_4(k) = \frac{kv_4}{4(k-1)}, \quad \{g_{4i}(k)\}_{i=0}^{4} = \{1,1,1,k^2,1\}.$$

Then $y_4(1) = y_4'(1) = y_4''(1) = (k^2y_4'')'(1) = 0$ and

$$(k^2y_4'')''(k) = \frac{2}{k^2}\left[(6k+21)\ln(k) + 22k^2 + 20k - 2\right] > 0, \quad k > 1.$$

Thus $y_4(k) > 0$ by Lemma 2.1, and hence $v_4(k) > 0$ for $k > 1$.

For $j = 5$, choose

$$y_5(k) = \frac{v_5(k)}{2k(k-1)}, \quad \{g_{5i}(k)\}_{i=0}^{3} = \{1,1,1,k^2\}.$$

Then $y_5(1) = y_5'(1) = y_5''(1) = 0$ and

$$(k^2y_5'')'(k) = \frac{1}{k}\left[21k\,(\ln(k))^2 + (200k - 102)\ln(k) + 224k^2 + 134k - 158\right] > 0, \quad k > 1.$$

It follows from Lemma 2.1 that $y_5(k) > 0$, and hence $v_5(k) > 0$ for $k > 1$.

For $j = 6$, choose

$$y_6(k) = \frac{v_6(k)}{4(k-1)}, \quad \{g_{6i}(k)\}_{i=0}^{3} = \{1,1,1,k^2\}.$$

Then $y_6(1) = y_6'(1) = 0$, $y_6''(1) = 125$, and

$$(k^2y_6'')'(k) = \frac{1}{k}\left[9k\,(\ln(k))^2 + (132k - 36)\ln(k) + 160k^2 + 231k - 54\right] > 0,\ k > 1.$$

From Lemma 2.1 we deduce that $y_6(k) > 0$ and, consequently, $v_6(k) > 0$ for $k > 1$. The proof has been completed.

Remark 1. In the above proof, we have shown that $U''/(U')^2$ is strictly decreasing, where $U(p) = \ln(M(p))$. This result is equivalent to the fact that the function

$$p \mapsto \frac{M(p)M''(p)}{(M'(p))^2} = \frac{U''(p)}{(U'(p))^2} + 1, \quad -\infty < p < \infty,$$

is strictly decreasing. It is actually stronger than the main assertion of the conjecture, which says that $M''(p) = 0$ has at most one solution.

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