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MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED USING CONVOLUTION

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ABSTRACT. For certain meromorphic function g and h, we study a class of functions $f(z) = z^{-1} + \sum_{n=1}^{\infty} f_n z^n$, $(f_n \geq 0)$, defined in the punctured unit disk Δ^*, satisfying

$$\Re\left(\frac{(f*g)(z)}{(f*h)(z)}\right) > \alpha \quad (z \in \Delta; 0 \leq \alpha < 1).$$

Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. Properties of an integral operator and its inverse defined on the new class is also discussed. In addition, we apply the concepts of neighborhoods of analytic functions to this class.

Key words and phrases: Meromorphic functions, Starlike function, Convolution, Positive coefficients, Coefficient inequalities, Growth and distortion theorems, Closure theorems, Integral operator.

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1. INTRODUCTION

Let Σ denote the class of normalized meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n, \tag{1.1}$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. A function $f \in \Sigma$ is *meromorphic starlike of order* α $(0 \leq \alpha < 1)$ if

$$-\Re \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \Delta := \Delta^* \cup \{0\}).$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$. Similarly the class of convex functions of order α is defined. Let Σ_p be the class of functions $f \in \Sigma$ with $f_n \geq 0$. The subclass of Σ_p consisting of starlike functions of order α is denoted by $\Sigma_p^*(\alpha)$. The following class $MR_p(\alpha)$ is related to the class of functions with positive real part:

$$MR_p(\alpha) := \left\{ f | \Re\{-z^2 f'(z)\} > \alpha, \ (0 \leq \alpha < 1) \right\}.$$

In Definition 1.1 below, we unify these classes by using convolution. The Hadamard product or convolution of two functions $f(z)$ given by (1.1) and

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} g_n z^n \tag{1.2}$$

is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} f_n g_n z^n.$$

Definition 1.1. Let $0 \leq \alpha < 1$. Let $f(z) \in \Sigma_p$ be given by (1.1) and $g(z) \in \Sigma_p$ be given by (1.2) and

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} h_n z^n. \tag{1.3}$$

Let h_n, g_n be real and $g_n + (1 - 2\alpha)h_n \leq 0 \leq \alpha h_n - g_n$. The class $M_p(g, h, \alpha)$ is defined by

$$M_p(g, h, \alpha) = \left\{ f \in \Sigma_p \middle| \Re \left(\frac{(f * g)(z)}{(f * h)(z)} \right) > \alpha \right\}.$$

Of course, one can consider a more general class of functions satisfying the subordination:

$$\frac{(f * g)(z)}{(f * h)(z)} \prec h(z) \quad (z \in \Delta).$$

However the results for this class will follow from the corresponding results of the class $M_p(g, h, \alpha)$. See [5] for details.

When

$$g(z) = \frac{1}{z} - \frac{z}{(1-z)^2} \quad \text{and} \quad h(z) = \frac{1}{z(1-z)},$$

we have $g_n = -n$ and $h_n = 1$ and therefore $M_p(g, h, \alpha)$ reduces to the class $\Sigma_p^*(\alpha)$. Similarly when

$$g(z) = \frac{1}{z} - \frac{z}{(1-z)^2} \quad \text{and} \quad h(z) = \frac{1}{z},$$

we have

$$M_p(g, h, \alpha) = \{f | -\Re\{z^2 f'(z)\} > \alpha\} =: MR_p(\alpha).$$

In this paper, coefficient inequalities, growth and distortion inequalities, as well as closure results for the class $M_p(g,h,\alpha)$ are obtained. Properties of an integral operator and its inverse defined on the new class $M_p(g,h,\alpha)$ is also discussed.

2. COEFFICIENTS INEQUALITIES

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $M_p(g,h,\alpha)$.

Theorem 2.1. *Let $f(z)\in\Sigma_p$ be given by (1.1). Then $f\in M_p(g,h,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}(\alpha h_n - g_n)f_n \le 1-\alpha. \tag{2.1}$$

Proof. If $f\in M_p(g,h,\alpha)$, then

$$\Re\left\{\frac{(f*g)(z)}{(f*h)(z)}\right\} = \Re\left\{\frac{1+\sum_{n=1}^{\infty} f_n g_n z^{n+1}}{1+\sum_{n=1}^{\infty} f_n h_n z^{n+1}}\right\} > \alpha.$$

By letting $z\to 1^-$, we have

$$\left\{\frac{1+\sum_{n=1}^{\infty} f_n g_n}{1+\sum_{n=1}^{\infty} f_n h_n}\right\} > \alpha.$$

This shows that (2.1) holds.

Conversely, assume that (2.1) holds. Since

$$\Re w > \alpha \quad \text{if and only if} \quad |w-1| < |w+1-2\alpha|,$$

it is sufficient to show that

$$\left|\frac{(f*g)(z)-(f*h)(z)}{(f*g)(z)+(1-2\alpha)(f*h)(z)}\right| < 1 \quad (z\in\Delta).$$

Using (2.1), we see that

$$\begin{aligned}\left|\frac{(f*g)(z)-(f*h)(z)}{(f*g)(z)+(1-2\alpha)(f*h)(z)}\right| &= \left|\frac{\sum_{n=1}^{\infty} f_n(g_n-h_n)z^{n+1}}{2(1-\alpha)+\sum_{n=1}^{\infty}[g_n+(1-2\alpha)h_n]f_n z^{n+1}}\right| \\ &\le \frac{\sum_{n=1}^{\infty} f_n(h_n-g_n)}{2(1-\alpha)-\sum_{n=1}^{\infty}[(2\alpha-1)h_n-g_n]f_n} \le 1.\end{aligned}$$

Thus we have $f\in M_p(g,h,\alpha)$. □

Corollary 2.2. *Let $f(z)\in\Sigma_p$ be given by (1.1). Then $f\in\Sigma_p^*(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}(n+\alpha)f_n \le 1-\alpha.$$

Corollary 2.3. *Let $f(z)\in\Sigma_p$ be given by (1.1). Then $f\in MR_p(\alpha)$ if and only if $\sum_{n=1}^{\infty} nf_n \le 1-\alpha$.*

Our next result gives the coefficient estimates for functions in $M_p(g,h,\alpha)$.

Theorem 2.4. *If $f\in M_p(g,h,\alpha)$, then*

$$f_n \le \frac{1-\alpha}{\alpha h_n - g_n}, \qquad n=1,2,3,\dots.$$

The result is sharp for the functions $F_n(z)$ given by

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_n - g_n}z^n, \qquad n=1,2,3,\dots.$$

Proof. If $f \in M_p(g,h,\alpha)$, then we have, for each n,

$$(\alpha h_n - g_n) f_n \leq \sum_{n=1}^{\infty} (\alpha h_n - g_n) f_n \leq 1-\alpha.$$

Therefore we have

$$f_n \leq \frac{1-\alpha}{\alpha h_n - g_n}.$$

Since

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_n - g_n} z^n$$

satisfies the conditions of Theorem 2.1, $F_n(z) \in M_p(g,h,\alpha)$ and the inequality is attained for this function. □

Corollary 2.5. *If $f \in \Sigma_p^*(\alpha)$, then*

$$f_n \leq \frac{1-\alpha}{n+\alpha}, \qquad n = 1,2,3,\dots.$$

Corollary 2.6. *If $f \in MR_p(\alpha)$, then*

$$f_n \leq \frac{1-\alpha}{n}, \qquad n = 1,2,3,\dots.$$

Theorem 2.7. *Let $\alpha h_1 - g_1 \leq \alpha h_n - g_n$. If $f \in M_p(g,h,\alpha)$, then*

$$\frac{1}{r} - \frac{1-\alpha}{\alpha h_1 - g_1} r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{\alpha h_1 - g_1} r \quad (|z| = r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_1 - g_1} z. \tag{2.2}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} f_n r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} f_n.$$

Since $\alpha h_1 - g_1 \leq \alpha h_n - g_n$, we have

$$(\alpha h_1 - g_1) \sum_{n=1}^{\infty} f_n \leq \sum_{n=1}^{\infty} (\alpha h_n - g_n) f_n \leq 1-\alpha,$$

and therefore

$$\sum_{n=1}^{\infty} f_n \leq \frac{1-\alpha}{\alpha h_1 - g_1}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{\alpha h_1 - g_1} r.$$

Similarly

$$|f(z)| \geq \frac{1}{r} - \frac{1-\alpha}{\alpha h_1 - g_1} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_1 - g_1} z$. □

Similarly we have the following:

Theorem 2.8. *Let* $\alpha h_1 - g_1 \leq (\alpha h_n - g_n)/n$. *If* $f \in M_p(g,h,\alpha)$, *then*

$$\frac{1}{r^2} - \frac{1-\alpha}{\alpha h_1 - g_1} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{\alpha h_1 - g_1} \quad (|z| = r).$$

The result is sharp for the function given by (2.2).

3. CLOSURE THEOREMS

Let the functions $F_k(z)$ be given by

$$F_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_{n,k} z^n, \quad k = 1, 2, \dots, m. \tag{3.1}$$

We shall prove the following closure theorems for the class $M_p(g,h,\alpha)$.

Theorem 3.1. *Let the function* $F_k(z)$ *defined by (3.1) be in the class* $M_p(g,h,\alpha)$ *for every* $k = 1, 2, \dots, m$. *Then the function* $f(z)$ *defined by*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_n \geq 0)$$

belongs to the class $M_p(g,h,\alpha)$, *where* $a_n = \frac{1}{m}\sum_{k=1}^{m} f_{n,k}$ $(n = 1, 2, \dots)$

Proof. Since $F_n(z) \in M_p(g,h,\alpha)$, it follows from Theorem 2.1 that

$$\sum_{n=1}^{\infty} (\alpha h_n - g_n) f_{n,k} \leq 1 - \alpha \tag{3.2}$$

for every $k = 1, 2, \dots, m$. Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} (\alpha h_n - g_n) a_n &= \sum_{n=1}^{\infty} (\alpha h_n - g_n) \left(\frac{1}{m} \sum_{k=1}^{m} f_{n,k} \right) \\
&= \frac{1}{m} \sum_{k=1}^{m} \left(\sum_{n=1}^{\infty} (\alpha h_n - g_n) f_{n,k} \right) \\
&\leq 1 - \alpha.
\end{aligned}$$

By Theorem 2.1, it follows that $f(z) \in M_p(g,h,\alpha)$. □

Theorem 3.2. *The class* $M_p(g,h,\alpha)$ *is closed under convex linear combination.*

Proof. Let the function $F_k(z)$ given by (3.1) be in the class $M_p(g,h,\alpha)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1-\lambda) F_2(z) \quad (0 \leq \lambda \leq 1)$$

is also in the class $M_p(g,h,\alpha)$. Since for $0 \leq \lambda \leq 1$,

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [\lambda f_{n,1} + (1-\lambda) f_{n,2}] z^n,$$

we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} (\alpha h_n - g_n)[\lambda f_{n,1} + (1-\lambda) f_{n,2}] &= \lambda \sum_{n=1}^{\infty} (\alpha h_n - g_n) f_{n,1} + (1-\lambda) \sum_{n=1}^{\infty} (\alpha h_n - g_n) f_{n,2} \\
&\leq 1 - \alpha.
\end{aligned}$$

By Theorem 2.1, we have $H(z) \in M_p(g,h,\alpha)$. □

Theorem 3.3. *Let $F_0(z) = \frac{1}{z}$ and $F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_n - g_n} z^n$ for $n = 1, 2, \ldots$. Then $f(z) \in M_p(g, h, \alpha)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.*

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda_n(1-\alpha)}{\alpha h_n - g_n} z^n.$$

Then

$$\sum_{n=1}^{\infty} \frac{\lambda_n(1-\alpha)}{\alpha h_n - g_n} \frac{\alpha h_n - g_n}{1-\alpha} = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1.$$

By Theorem 2.1, we have $f(z) \in M_p(g, h, \alpha)$.

Conversely, let $f(z) \in M_p(g, h, \alpha)$. From Theorem 2.4, we have

$$f_n \leq \frac{1-\alpha}{\alpha h_n - g_n} \quad \text{for} \quad n = 1, 2, \ldots$$

we may take

$$\lambda_n = \frac{\alpha h_n - g_n}{1-\alpha} f_n \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n.$$

Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z).$$

□

4. INTEGRAL OPERATORS

In this section, we consider integral transforms of functions in the class $M_p(g, h, \alpha)$.

Theorem 4.1. *Let the function $f(z)$ given by (1.1) be in $M_p(g, h, \alpha)$. Then the integral operator*

$$F(z) = c \int_0^1 u^c f(uz) du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $M_p(g, h, \delta)$, where

$$\delta = \frac{(c+2)(\alpha h_1 - g_1) + (1-\alpha) c g_1}{(c+2)(\alpha h_1 - g_1) + (1-\alpha) c h_1}.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_1 - g_1} z$.

Proof. Let $f(z) \in M_p(g, h, \alpha)$. Then

$$\begin{aligned} F(z) &= c \int_0^1 u^c f(uz) du \\ &= c \int_0^1 \left(\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} f_n u^{n+c} z^n \right) du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} f_n z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c(\delta h_n - g_n)}{(c+n+1)(1-\delta)} f_n \leq 1. \tag{4.1}$$

Since $f \in M_p(g,h,\alpha)$, we have

$$\sum_{n=1}^{\infty} \frac{\alpha h_n - g_n}{(1-\alpha)} f_n \leq 1.$$

Note that (4.1) is satisfied if

$$\frac{c(\delta h_n - g_n)}{(c+n+1)(1-\delta)} \leq \frac{\alpha h_n - g_n}{(1-\alpha)}.$$

Rewriting the inequality, we have

$$c(\delta h_n - g_n)(1-\alpha) \leq (c+n+1)(1-\delta)(\alpha h_n - g_n).$$

Solving for δ, we have

$$\delta \leq \frac{(\alpha h_n - g_n)(c+n+1) + c g_n(1-\alpha)}{c h_n(1-\alpha) + (\alpha h_n - g_n)(c+n+1)} = F(n).$$

A computation shows that

$$\begin{aligned} &F(n+1) - F(n) \\ &= \frac{(1-\alpha)c[(1-\alpha)(n+1)g_n h_{n+1} + (h_n - g_n)(\alpha h_{n+1} - g_{n+1})]}{[c h_n(1-\alpha) + (\alpha h_n - g_n)(c+n+1)][c h_{n+1}(1-\alpha) + (\alpha h_{n+1} - g_{n+1})(c+n+2)]} > 0 \end{aligned}$$

for all n. This means that $F(n)$ is increasing and $F(n) \geq F(1)$. Using this, the results follows. □

In particular, we have the following result of Uralegaddi and Ganigi [4]:

Corollary 4.2. *Let the function $f(z)$ defined by (1.1) be in $\Sigma_p^*(\alpha)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $\Sigma_p^*(\delta)$, where $\delta = \frac{1+\alpha+c\alpha}{1+\alpha+c}$. The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha} z.$$

Also we have the following:

Corollary 4.3. *Let the function $f(z)$ defined by (1.1) be in $MR_p(\alpha)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $MR_p(\frac{2+c\alpha}{c+2})$. The result is sharp for the function $f(z) = \frac{1}{z} + (1-\alpha)z$.

Theorem 4.4. *Let $f(z)$, given by (1.1), be in $M_p(g,h,\alpha)$,*

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c+n+1}{c} f_n z^n, \qquad c > 0. \tag{4.2}$$

Then $F(z)$ is in $M_p(g,h,\beta)$ for $|z| \leq r(\alpha,\beta)$, where

$$r(\alpha,\beta) = \inf_n \left(\frac{c(1-\beta)(\alpha h_n - g_n)}{(1-\alpha)(c+n+1)(\beta h_n - g_n)} \right)^{\frac{1}{n+1}}, \quad n = 1,2,3,\dots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\alpha}{\alpha h_n - g_n} z^n, \quad n = 1,2,3,\dots.$

Proof. Let $w = \frac{(f*g)(z)}{(f*h)(z)}$. Then it is sufficient to show that

$$\left| \frac{w-1}{w+1-2\beta} \right| < 1.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{(\beta h_n - g_n)(c+n+1)}{(1-\beta)c} f_n |z|^{n+1} \leq 1. \tag{4.3}$$

Since $f \in M_p(g,h,\alpha)$, by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} (\alpha h_n - g_n) f_n \leq 1 - \alpha.$$

The equation (4.3) is satisfied if

$$\frac{(\beta h_n - g_n)(c+n+1)}{(1-\beta)c} f_n |z|^{n+1} \leq \frac{(\alpha h_n - g_n) f_n}{1-\alpha}.$$

Solving for $|z|$, we get the result. □

In particular, we have the following result of Uralegaddi and Ganigi [4]:

Corollary 4.5. *Let the function $f(z)$ defined by (1.1) be in $\Sigma_p^*(\alpha)$ and $F(z)$ given by (4.2). Then $F(z)$ is in $\Sigma_p^*(\alpha)$ for $|z| \leq r(\alpha,\beta)$, where*

$$r(\alpha,\beta) = \inf_n \left(\frac{c(1-\beta)(n+\alpha)}{(1-\alpha)(c+n+1)(n+\beta)} \right)^{\frac{1}{n+1}}, \quad n = 1,2,3,\dots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n+\alpha} z^n, \quad n = 1,2,3,\dots.$

Corollary 4.6. *Let the function $f(z)$ defined by (1.1) be in $MR_p(\alpha)$ and $F(z)$ given by (4.2). Then $F(z)$ is in $MR_p(\alpha)$ for $|z| \leq r(\alpha,\beta)$, where*

$$r(\alpha,\beta) = \inf_n \left(\frac{c(1-\beta)}{(1-\alpha)(c+n+1)} \right)^{\frac{1}{n+1}}, \quad n = 1,2,3,\dots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n} z^n, \quad n = 1,2,3,\dots.$

5. NEIGHBORHOODS FOR THE CLASS $M_p^{(\gamma)}(g,h,\alpha)$

In this section, we determine the neighborhood for the class $M_p^{(\gamma)}(g,h,\alpha)$, which we define as follows:

Definition 5.1. A function $f \in \Sigma_p$ is said to be in the class $M_p^{(\gamma)}(g,h,\alpha)$ if there exists a function $g \in M_p(g,h,\alpha)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \gamma, \qquad (z \in \Delta, 0 \leq \gamma < 1). \tag{5.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [1] and Ruscheweyh [3], we define the δ-neighborhood of a function $f \in \Sigma_p$ by

$$N_\delta(f) := \left\{ g \in \Sigma_p \ : \ g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{5.2}$$

Theorem 5.1. *If* $g \in M_p(g, h, \alpha)$ *and*

$$\gamma = 1 - \frac{\delta(\alpha h_1 - g_1)}{\alpha(h_1 + 1) - (g_1 + 1)}, \tag{5.3}$$

then

$$N_\delta(g) \subset M_p^{(\gamma)}(g, h, \alpha).$$

Proof. Let $f \in N_\delta(g)$. Then we find from (5.2) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \tag{5.4}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \qquad (n \in \mathbb{N}). \tag{5.5}$$

Since $g \in M_p(g, h, \alpha)$, we have [cf. equation (2.1)]

$$\sum_{n=1}^{\infty} f_n \leq \frac{1 - \alpha}{(\alpha h_1 - g_1)}, \tag{5.6}$$

so that

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \\ &= \frac{\delta(\alpha h_1 - g_1)}{\alpha(h_1 + 1) - (g_1 + 1)} \\ &= 1 - \gamma, \end{aligned}$$

provided γ is given by (5.3). Hence, by definition, $f \in M_p^{(\gamma)}(g, h, \alpha)$ for γ given by (5.3), which completes the proof. $\square$

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