

MULTIPLICATION OF SUBHARMONIC FUNCTIONS

R. SUPPER

Université Louis Pasteur

UFR de Mathématique et Informatique, URA CNRS 001

7 rue René Descartes

F-67 084 Strasbourg Cedex, France

E-Mail: supper@math.u-strasbg.fr

Received: 31 May, 2008

Accepted: 12 November, 2008

Communicated by: [S.S. Dragomir](#)

2000 AMS Sub. Class.: 31B05, 33B15, 26D15, 30D45.

Key words: Gamma and Beta functions, growth of subharmonic functions in the unit ball.

Abstract: We study subharmonic functions in the unit ball of \mathbb{R}^N , with either a Bloch-type growth or a growth described through integral conditions involving some involutions of the ball. Considering mappings $u \mapsto gu$ between sets of functions with a prescribed growth, we study how the choice of these sets is related to the growth of the function g .



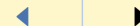
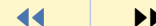
**Multiplication of
Subharmonic Functions**

R. Supper

vol. 9, iss. 4, art. 118, 2008

[Title Page](#)

[Contents](#)



Page **1** of 40

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents



Page 2 of 40

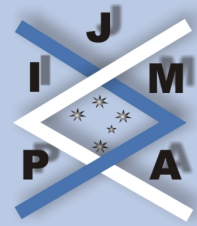
Go Back

Full Screen

Close

Contents

1	Introduction	3
2	Notations and Main Results	4
3	Some Preliminaries	7
4	Proof of Theorem 2.2	14
4.1	Proof of Theorem 2.2 in the case (i)	22
4.2	Proof of Theorem 2.2 in the case (ii)	24
4.3	Proof of Theorem 2.2 in the case (iii)	25
4.4	Proof of Theorem 2.2 in the case (iv)	27
4.5	Proof of Theorem 2.2 in the case (v)	28
4.6	Proof of Theorem 2.2 in the case (vi)	29
5	The Situation with Radial Subharmonic Functions	30
5.1	The example of $u : x \mapsto (1 - x ^2)^{-A}$ with $A \geq 0$	30
5.2	Proof of Theorem 2.3	33
6	Annex: The Sets $\mathcal{S}\mathcal{X}_\lambda$ and $\mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma}$ for some Special Values of $\lambda, \alpha, \beta, \gamma$	35



Title Page

Contents



Page 3 of 40

Go Back

Full Screen

Close

1. Introduction

This paper is devoted to functions u which are defined in the unit ball B_N of \mathbb{R}^N (relative to the Euclidean norm $|\cdot|$), whose growth is described by the above boundedness on B_N of $x \mapsto (1 - |x|^2)^\alpha v(x)$ for some parameter α . The function v may denote merely u or some integral involving u and involutions Φ_x (precise definitions and notations will be detailed in Section 2). In the first (resp. second) case, u is said to belong to the set \mathcal{X} (resp. \mathcal{Y}). Given a function g defined on B_N , we try to obtain links between the growth of g and information on such mappings as

$$\begin{aligned}\mathcal{Y} &\rightarrow \mathcal{X}, \\ u &\mapsto gu.\end{aligned}$$

This work is motivated by the situation known in the case of holomorphic functions f in the unit disk D of \mathbb{C} . Such a function is said to belong to the Bloch space \mathcal{B}_λ if

$$\|f\|_{\mathcal{B}_\lambda} := |f(0)| + \sup_{z \in D} (1 - |z|^2)^\lambda |f'(z)| < +\infty.$$

It is said to belong to the space $BMOA_\mu$ if

$$\|f\|_{BMOA_\mu}^2 := |f(0)|^2 + \sup_{a \in D} \int_D (1 - |z|^2)^{2\mu-2} |f'(z)|^2 (1 - |\varphi_a(z)|^2) dA(z) < +\infty$$

with $dA(z)$ the normalized area measure element on D and $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Given h a holomorphic function on D , the operator $I_h : f \mapsto I_h(f)$ defined by:

$$(I_h(f))(z) = \int_0^z h(\zeta) f'(\zeta) d\zeta \quad \forall z \in D$$

was studied for instance in [7] where it was proved that $I_h : BMOA_\mu \rightarrow \mathcal{B}_\lambda$ is bounded (with respect to the above norms) if and only if $h \in \mathcal{B}_{\lambda-\mu+1}$ (assuming $1 < \mu < \lambda$).

Since $|f'|^2$ is subharmonic in the unit ball of \mathbb{R}^2 , the question naturally arose whether some similar phenomena occur for subharmonic functions in B_N for $N \geq 2$.



2. Notations and Main Results

Let $B_N = \{x \in \mathbb{R}^N : |x| < 1\}$ with $N \in \mathbb{N}$, $N \geq 2$ and $|\cdot|$ the Euclidean norm in \mathbb{R}^N . Given $a \in B_N$, let $\Phi_a : B_N \rightarrow B_N$ denote the involution defined by:

$$\Phi_a(x) = \frac{a - P_a(x) - \sqrt{1 - |a|^2} Q_a(x)}{1 - \langle x, a \rangle} \quad \forall x \in B_N,$$

where

$$\langle x, a \rangle = \sum_{j=1}^N x_j a_j, \quad P_a(x) = \frac{\langle x, a \rangle}{|a|^2} a, \quad Q_a(x) = x - P_a(x)$$

for all $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ and $a = (a_1, a_2, \dots, a_N) \in \mathbb{R}^N$, with $P_a(x) = 0$ if $a = 0$. We refer to [4, pp. 25–26] and [1, p. 115] for the main properties of the map Φ_a (initially defined in the unit ball of \mathbb{C}^N). For instance, we will make use of the relation:

$$1 - |\Phi_a(x)|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{(1 - \langle x, a \rangle)^2}.$$

In the following, α, β, γ and λ are given real numbers, with $\gamma \geq 0$.

Definition 2.1. Let \mathcal{X}_λ denote the set of all functions $u : B_N \rightarrow [-\infty, +\infty]$ satisfying:

$$M_{\mathcal{X}_\lambda}(u) := \sup_{x \in B_N} (1 - |x|^2)^\lambda u(x) < +\infty.$$

Let $\mathcal{Y}_{\alpha, \beta, \gamma}$ denote the set of all measurable functions $u : B_N \rightarrow [-\infty, +\infty]$ satisfying:

$$M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(u) := \sup_{a \in B_N} (1 - |a|^2)^\alpha \int_{B_N} (1 - |x|^2)^\beta u(x) (1 - |\Phi_a(x)|^2)^\gamma dx < +\infty.$$



Title Page

Contents



Page 5 of 40

Go Back

Full Screen

Close

The subset \mathcal{SX}_λ (resp. $\mathcal{SY}_{\alpha,\beta,\gamma}$) gathers all $u \in \mathcal{X}_\lambda$ (resp. $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$) which moreover are subharmonic and non-negative. The subset $\mathcal{RSY}_{\alpha,\beta,\gamma}$ gathers all $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$ which moreover are radial.

Remark 1. When $\lambda < 0$ (resp. $\alpha + \beta < -N$ or $\alpha < -\gamma$), the set \mathcal{SX}_λ (resp. $\mathcal{SY}_{\alpha,\beta,\gamma}$) merely reduces to the single function $u \equiv 0$ (see Propositions 6.2, 6.3 and 6.4).

In Proposition 3.1 and Corollary 3.2, we will establish that $\mathcal{SY}_{\alpha,\beta,\gamma} \subset \mathcal{SX}_{\alpha+\beta+N}$ and that there exists a constant $C > 0$ such that

$$M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C M_{\mathcal{X}_\lambda}(g) M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)$$

for all $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$ and all $g \in \mathcal{X}_\lambda$ with $M_{\mathcal{X}_\lambda}(g) \geq 0$. We will next study whether some kind of a “converse” holds and obtain the following:

Theorem 2.2. Given $\lambda \in \mathbb{R}$ and $g : B_N \rightarrow [0, +\infty[$ a subharmonic function satisfying:

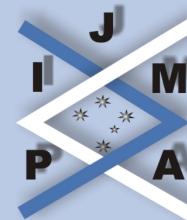
$$\exists C' > 0 \quad M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C' M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \quad \forall u \in \mathcal{SY}_{\alpha,\beta,\gamma},$$

then $g \in \mathcal{X}_{\lambda+\frac{N-1}{2}}$ in each of the six cases gathered in the following Table 1.

Theorem 2.3. Given $\lambda \in \mathbb{R}$ and g a subharmonic function defined on B_N , satisfying:

$$\exists C'' > 0 \quad M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C'' M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \quad \forall u \in \mathcal{RSY}_{\alpha,\beta,\gamma},$$

then $g \in \mathcal{SX}_{\lambda+\alpha+\frac{N-1}{2}}$ provided that $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$.



case	α	β	γ
(i)	$\alpha = \frac{N+1}{2} + \beta$	$\beta > -\frac{N+1}{2}$	$\gamma > \max(\alpha, -1 - \beta)$
(ii)	$\alpha = \beta + 1$	$\beta > -\frac{N+3}{4}$	$\gamma > 1 + \beta $
(iii)	$\alpha = \frac{N+1}{2} - \gamma$	$\beta \geq -\gamma$	$\frac{N+1}{4} < \gamma < \frac{N+1}{2}$
(iv)	$\alpha = 1$	$\beta \geq 0$	$\gamma > 1$
(v)	$\alpha = 1 + \beta - \gamma$	$\beta > -1$	$\frac{1+\beta}{2} < \gamma < \beta + \frac{N+3}{4}$
(vi)	$\alpha = \frac{\beta+1}{2}$	$\beta \geq -\frac{1}{2}$	$\gamma > \left \frac{1+\beta}{2} \right $

Table 1: Six situations where Theorem 2.2 shows that g belongs to the set $\mathcal{X}_{\lambda + \frac{N-1}{2}}$.

Title Page

Contents

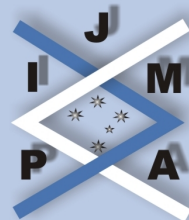


Page 6 of 40

Go Back

Full Screen

Close



3. Some Preliminaries

Notation 1. Given $a \in B_N$ and $R \in]0, 1[$, let $B(a, R_a) = \{x \in B_N : |x - a| < R_a\}$ with

$$R_a = R \frac{1 - |a|^2}{1 + R|a|}.$$

Proposition 3.1. *There exists a $C > 0$ depending only on N, β, γ , such that:*

$$M_{\mathcal{X}_{\alpha+\beta+N}}(u) \leq C M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \quad \forall u \in \mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma}.$$

Proof. Let some $R \in]0, 1[$ be fixed in the following. Since $u \geq 0$, we obtain for any $a \in B_N$:

$$\begin{aligned} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) &\geq (1 - |a|^2)^\alpha \int_{B_N} (1 - |x|^2)^\beta u(x) (1 - |\Phi_a(x)|^2)^\gamma dx \\ &\geq (1 - |a|^2)^\alpha \int_{B(a,R_a)} (1 - |x|^2)^\beta u(x) (1 - |\Phi_a(x)|^2)^\gamma dx. \end{aligned}$$

It follows from Lemma 1 of [6] that

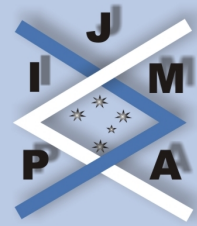
$$B(a, R_a) \subset E(a, R) = \{x \in B_N : |\Phi_a(x)| < R\},$$

hence:

$$(3.1) \quad M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq (1 - R^2)^\gamma (1 - |a|^2)^\alpha \int_{B(a,R_a)} (1 - |x|^2)^\beta u(x) dx$$

as $\gamma \geq 0$. From Lemmas 1 and 5 of [5], it is known that

$$\frac{1 - R}{1 + R} \leq \frac{1 - |x|^2}{1 - |a|^2} \leq 2 \quad \forall x \in B(a, R_a).$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 40

Go Back

Full Screen

Close

Let $C_\beta = \left(\frac{1-R}{1+R}\right)^\beta$ if $\beta \geq 0$ and $C_\beta = 2^\beta$ if $\beta < 0$. Hence

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq C_\beta(1-R^2)^\gamma(1-|a|^2)^{\alpha+\beta} \int_{B(a,R_a)} u(x) dx.$$

The volume of $B(a, R_a)$ is $\sigma_N \frac{(R_a)^N}{N}$ with $\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ the area of the unit sphere S_N in \mathbb{R}^N (see [2, p. 29]) and $R_a \geq \frac{R}{1+R}(1-|a|^2)$. The subharmonicity of u now provides:

$$\begin{aligned} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) &\geq C_\beta(1-R^2)^\gamma(1-|a|^2)^{\alpha+\beta} u(a) \sigma_N \frac{(R_a)^N}{N} \\ &\geq C_\beta \frac{\sigma_N}{N} \frac{R^N(1-R)^\gamma}{(1+R)^{N-\gamma}} (1-|a|^2)^{\alpha+\beta+N} u(a). \end{aligned}$$

□

Corollary 3.2. Let $g \in \mathcal{X}_\lambda$ with $M_{\mathcal{X}_\lambda}(g) \geq 0$. Then:

$$M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C M_{\mathcal{X}_\lambda}(g) M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \quad \forall u \in \mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma}$$

where the constant C stems from Proposition 3.1.

Proof. Since $u \geq 0$, we have for any $x \in B_N$:

$$\begin{aligned} (1-|x|^2)^{\lambda+\alpha+\beta+N} g(x) u(x) &\leq M_{\mathcal{X}_\lambda}(g) (1-|x|^2)^{\alpha+\beta+N} u(x) \\ &\leq M_{\mathcal{X}_\lambda}(g) M_{\mathcal{X}_{\alpha+\beta+N}}(u) \end{aligned}$$

because of $M_{\mathcal{X}_\lambda}(g) \geq 0$. □

Lemma 3.3. Given $a \in B_N$ and $R \in]0, 1[$, the following holds for any $x \in B(a, R_a)$:

$$\frac{1}{2} < \frac{1}{1+R|a|} \leq \frac{1-\langle x, a \rangle}{1-|a|^2} \leq \frac{1+2R|a|}{1+R|a|} < 2 \quad \text{and} \quad \frac{1}{4} < \frac{1-\langle x, a \rangle}{1-|x|^2} < 2 \frac{1+R}{1-R}.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 9 of 40

Go Back

Full Screen

Close

Proof. Clearly $\langle x, a \rangle = \langle a + y, a \rangle = |a|^2 + \langle y, a \rangle$ with $|y| < R_a$. From the Cauchy-Schwarz inequality, it follows that $-R_a |a| \leq \langle y, a \rangle \leq R_a |a|$. Hence:

$$1 - |a|^2 - R |a| \frac{1 - |a|^2}{1 + R|a|} \leq 1 - \langle x, a \rangle \leq 1 - |a|^2 + R |a| \frac{1 - |a|^2}{1 + R|a|}.$$

The term on the left equals

$$(1 - |a|^2) \left(1 - \frac{R |a|}{1 + R|a|} \right) = (1 - |a|^2) \frac{1}{1 + R|a|}$$

and $1 + R|a| < 2$. The term on the right equals

$$(1 - |a|^2) \left(1 + \frac{R |a|}{1 + R|a|} \right),$$

with $\frac{R|a|}{1+R|a|} < 1$. Now

$$\frac{1 - \langle x, a \rangle}{1 - |x|^2} = \frac{1 - \langle x, a \rangle}{1 - |a|^2} \frac{1 - |a|^2}{1 - |x|^2}$$

and the last inequalities follow from Lemmas 1 and 5 of [5]. □

Lemma 3.4. Let $H = \{(s, t) \in \mathbb{R}^2 : t \geq 0, s^2 + t^2 < 1\}$ and $P > -1, Q > -1, T > -1$. Then

$$\iint_H s^P t^Q (1 - s^2 - t^2)^T ds dt = \begin{cases} 0 & \text{if } P \text{ is odd;} \\ \frac{\Gamma(\frac{P+1}{2})\Gamma(\frac{Q+1}{2})\Gamma(T+1)}{2\Gamma(\frac{P+Q}{2}+T+2)} & \text{if } P \text{ is even.} \end{cases}$$



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 10 of 40

[Go Back](#)

[Full Screen](#)

[Close](#)

Proof. With polar coordinates $s = r \cos \theta$, $t = r \sin \theta$, this integral turns into $I_1 I_2$ with

$$I_1 = \int_0^1 r^{P+Q} (1-r^2)^T r dr \quad \text{and} \quad I_2 = \int_0^\pi (\cos \theta)^P (\sin \theta)^Q d\theta.$$

Keeping in mind the various expressions for the Beta function (see [3, pp. 67–68]):

$$\begin{aligned} B(x, y) &= \int_0^1 \xi^{x-1} (1-\xi)^{y-1} d\xi \\ &= 2 \int_0^{\pi/2} (\cos \theta)^{2x-1} (\sin \theta)^{2y-1} d\theta = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

(with $x > 0$ and $y > 0$), the change of variable $\omega = r^2$ leads to:

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^1 \omega^{\frac{P+Q}{2}} (1-\omega)^T d\omega \\ &= \frac{1}{2} B\left(\frac{P+Q}{2} + 1, T + 1\right) = \frac{\Gamma\left(\frac{P+Q}{2} + 1\right) \Gamma(T + 1)}{2 \Gamma\left(\frac{P+Q}{2} + T + 2\right)}. \end{aligned}$$

When P is odd, $I_2 = 0$ because $\cos(\pi - \theta) = -\cos(\theta)$. However, when P is even:

$$\begin{aligned} I_2 &= 2 \int_0^{\pi/2} (\cos \theta)^P (\sin \theta)^Q d\theta \\ &= B\left(\frac{P+1}{2}, \frac{Q+1}{2}\right) = \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{Q+1}{2}\right)}{\Gamma\left(\frac{P+Q}{2} + 1\right)}. \end{aligned}$$

□



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 11 of 40

[Go Back](#)

[Full Screen](#)

[Close](#)

Lemma 3.5. Given $A \geq 0$ and $a \in B_N$, let u and f_a denote the functions defined on B_N by $u(x) = \frac{1}{(1-|x|^2)^A}$ and $f_a(x) = \frac{1}{(1-\langle x, a \rangle)^A} \forall x \in B_N$. They are both subharmonic in B_N .

Remark 2. u is radial, but not f_a .

Proof. For u , the result of Lemma 3.5 has already been proved in Proposition 1 of [5]. For any $j \in \{1, 2, \dots, N\}$, we now compute:

$$\frac{\partial f_a}{\partial x_j}(x) = a_j A (1 - \langle x, a \rangle)^{-A-1} \quad \text{and} \quad \frac{\partial^2 f_a}{\partial x_j^2}(x) = (a_j)^2 A (A+1) (1 - \langle x, a \rangle)^{-A-2},$$

so that:

$$(\Delta f_a)(x) = \frac{|a|^2 A (A+1)}{(1 - \langle x, a \rangle)^{A+2}} \geq 0 \quad \forall x \in B_N.$$

□

Remark 3. Given $A \geq 0$, $A' \geq 0$, the function f_a defined on B_N by

$$f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^A (1 - |x|^2)^{A'}}$$

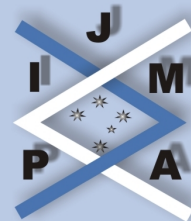
is subharmonic too. The computation

$$(\Delta f_a)(x) \geq f_a(x) \left(\frac{A|a|^2}{1 - \langle x, a \rangle} - \frac{2A'|x|^2}{1 - |x|^2} \right)^2 \geq 0$$

is left to the reader.

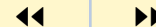
Proposition 3.6. Given $N \in \mathbb{N}$, $N > 3$, $(s, t, b_1, b_2) \in \mathbb{R}^4$ such that $|s b_1| + |t b_2| < 1$ and $P > 0$, let

$$I_P(s, t, b_1, b_2) = \int_0^\pi \frac{(\sin \theta)^{N-3} d\theta}{(1 - s b_1 - t b_2 \cos \theta)^P}.$$



Title Page

Contents



Page 12 of 40

Go Back

Full Screen

Close

Then

$$I_P(s, t, b_1, b_2) = \sqrt{\pi} \frac{\Gamma\left(\frac{N}{2} - 1\right)}{\Gamma(P)} \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} \frac{\Gamma(j + 2k + P)}{k! j! \Gamma\left(\frac{N-1}{2} + k\right)} (b_1 s)^j \left(\frac{t b_2}{2}\right)^{2k}.$$

Proof. As

$$\left| \frac{t b_2 \cos \theta}{1 - s b_1} \right| \leq \left| \frac{t b_2}{1 - s b_1} \right| < 1,$$

the following development is valid:

$$\begin{aligned} I_P(s, t, b_1, b_2) &= \int_0^\pi \frac{(\sin \theta)^{N-3} d\theta}{(1 - s b_1)^P \left(1 - \frac{t b_2 \cos \theta}{1 - s b_1}\right)^P} \\ &= \frac{1}{(1 - s b_1)^P} \sum_{n \in \mathbb{N}} \frac{\Gamma(n + P)}{n! \Gamma(P)} \left(\frac{t b_2}{1 - s b_1}\right)^n \int_0^\pi (\sin \theta)^{N-3} (\cos \theta)^n d\theta. \end{aligned}$$

The last integral vanishes when n is odd. When n is even ($n = 2k$), then

$$\begin{aligned} 2 \int_0^{\pi/2} (\sin \theta)^{N-3} (\cos \theta)^{2k} d\theta &= B\left(\frac{N-2}{2}, k + \frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{N-2}{2}\right) \Gamma\left(k + \frac{1}{2}\right)}{\Gamma\left(\frac{N-1}{2} + k\right)} \\ &= \frac{\Gamma\left(\frac{N-2}{2}\right) (2k)! \sqrt{\pi}}{\Gamma\left(\frac{N-1}{2} + k\right) 2^{2k} k!} \end{aligned}$$

by [3, p. 40]. Hence:

$$I_P(s, t, b_1, b_2) = \frac{\Gamma\left(\frac{N-2}{2}\right) \sqrt{\pi}}{\Gamma(P)} \sum_{k \in \mathbb{N}} \frac{\Gamma(2k + P)}{\Gamma\left(\frac{N-1}{2} + k\right) 2^{2k} k!} \frac{(t b_2)^{2k}}{(1 - s b_1)^{2k+P}}.$$

The result follows from the expansion

$$\frac{\Gamma(2k + P)}{(1 - s b_1)^{2k+P}} = \sum_{j \in \mathbb{N}} \frac{\Gamma(j + 2k + P)}{j!} (b_1 s)^j.$$

□



**Multiplication of
Subharmonic Functions**

R. Supper

vol. 9, iss. 4, art. 118, 2008

Title Page

Contents



Page 13 of 40

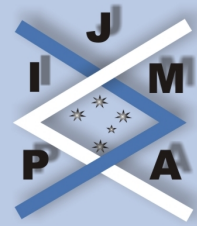
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



4. Proof of Theorem 2.2

The cases (i), (ii), (iii), (iv), (v) and (vi) of Theorem 2.2 will be proved separately at the end of this section.

Theorem 4.1. *Given $A > 0$, $P > 0$, $T > -1$ and $N \in \mathbb{N}$ ($N \geq 2$) such that $1 \leq A + P \leq N + 1 + 2T$, let*

$$I_{A,P,T}(a, b) = \int_{B_N} \frac{(1 - |x|^2)^T}{(1 - \langle x, a \rangle)^A (1 - \langle x, b \rangle)^P} dx \quad \forall a \in B_N, \forall b \in B_N$$

and τ a number satisfying both $\frac{P-A}{2} < \tau < P$ and $0 \leq \tau \leq \frac{A+P}{2}$. Then

$$I_{A,P,T}(a, b) \leq \frac{K}{(1 - |a|^2)^{\frac{A+P}{2}-\tau} (1 - |b|^2)^\tau} \quad \forall a \in B_N, \forall b \in B_N$$

where the constant K is independent of a and b .

Example 4.1. If $P > A$ and $\tau = \frac{A+P}{2}$, then

$$I_{A,P,T}(a, b) \leq \frac{K}{(1 - |b|^2)^{\frac{A+P}{2}}} \quad \forall a \in B_N, \forall b \in B_N,$$

with

$$K = 2^{A+P-1} \pi^{\frac{N-1}{2}} \frac{\Gamma(T+1)}{\Gamma(P)} \Gamma\left(\frac{P-A}{2}\right).$$

Example 4.2. If $P < A$ and $\tau = 0$, then

$$I_{A,P,T}(a, b) \leq \frac{K}{(1 - |a|^2)^{\frac{A+P}{2}}} \quad \forall a \in B_N, \forall b \in B_N,$$



Title Page

Contents



Page 15 of 40

Go Back

Full Screen

Close

with

$$K = 2^{A+P-1} \pi^{\frac{N-1}{2}} \frac{\Gamma(T+1)}{\Gamma(A)} \Gamma\left(\frac{A-P}{2}\right).$$

Proof. Up to a unitary transform, we assume $a = (|a|, 0, 0, \dots, 0)$ and $b = (b_1, b_2, 0, \dots, 0)$.

Proof of Theorem 4.1 in the case $N > 3$. Polar coordinates in \mathbb{R}^N provide the formulas: $x_1 = r \cos \theta_1$ with $r = |x|$, $x_2 = r \sin \theta_1 \cos \theta_2$ (the formulas for x_3, \dots, x_N are available in [9, p. 15]) where $\theta_1, \theta_2, \dots, \theta_{N-2} \in]0, \pi[$ and $\theta_{N-1} \in]0, 2\pi[$. The volume element dx becomes $r^{N-1} dr d\sigma^{(N)}$ where $d\sigma^{(N)}$ denotes the area element on S_N , with

$$d\sigma^{(N)} = (\sin \theta_1)^{N-2} (\sin \theta_2)^{N-3} d\theta_1 d\theta_2 d\sigma^{(N-2)}$$

(see [9, p. 15] for full details). Here $\theta_2 \in]0, \pi[$ since $N > 3$. In the following, we will write $s = r \cos \theta_1$ and $t = r \sin \theta_1$, thus $\langle x, b \rangle = s b_1 + t b_2 \cos \theta_2$ and

$$(4.1) \quad I_{A,P,T}(a, b) = \sigma_{N-2} \int_0^\pi \int_0^1 \frac{(1-r^2)^T r^{N-1} (\sin \theta_1)^{N-2} I_P(s, t, b_1, b_2)}{(1-|a|s)^A} dr d\theta_1$$

with $I_P(s, t, b_1, b_2)$ defined in the previous proposition. From [2, p. 29] we notice that

$$\sigma_{N-2} \Gamma\left(\frac{N-2}{2}\right) \sqrt{\pi} = 2\pi^{\frac{N-1}{2}}.$$

The expansion

$$\frac{1}{(1-|a|s)^A} = \sum_{\ell \in \mathbb{N}} \frac{\Gamma(\ell+A)}{\ell! \Gamma(A)} (|a|s)^\ell$$

leads to:

$$I_{A,P,T}(a, b) = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(P)\Gamma(A)} \sum_{(k,j,\ell) \in \mathbb{N}^3} \frac{\Gamma(j+2k+P)\Gamma(\ell+A)}{k! j! \ell! \Gamma(\frac{N-1}{2}+k)} (b_1)^j \left(\frac{b_2}{2}\right)^{2k} |a|^\ell J_{k,j,\ell}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 16 of 40

Go Back

Full Screen

Close

where

$$\begin{aligned} J_{k,j,\ell} &= \int_0^\pi \int_0^1 s^{j+\ell} t^{2k} (1-r^2)^T r^{N-1} (\sin \theta_1)^{N-2} dr d\theta_1 \\ &= \iint_H s^{j+\ell} t^{2k+N-2} (1-s^2-t^2)^T ds dt \end{aligned}$$

with H as in Lemma 3.4. Now $J_{k,j,\ell} = 0$ unless $j + \ell = 2h$ ($h \in \mathbb{N}$). Thus:

$$\begin{aligned} I_{A,P,T}(a,b) &= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(P)\Gamma(A)} \\ &\times \sum_{(k,h) \in \mathbb{N}^2} \sum_{j=0}^{2h} \frac{\Gamma(j+2k+P)\Gamma(2h-j+A)\Gamma(h+\frac{1}{2})\Gamma(T+1)}{k!j!(2h-j)!\Gamma(k+h+\frac{N}{2}+T+1)} (b_1)^j \left(\frac{b_2}{2}\right)^{2k} |a|^{2h-j} \end{aligned}$$

Taking [3, p. 40] into account:

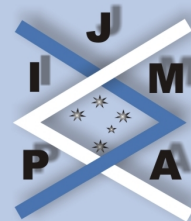
$$\begin{aligned} (4.2) \quad I_{A,P,T}(a,b) &= \frac{\pi^{\frac{N}{2}}\Gamma(T+1)}{\Gamma(P)\Gamma(A)} \sum_{(k,h) \in \mathbb{N}^2} \sum_{j=0}^{2h} \frac{(2h)!B(j+2k+P,2h-j+A)}{2^{2h+2k}h!k!j!(2h-j)!} \\ &\times \frac{\Gamma(2k+P+2h+A)}{\Gamma(k+h+\frac{N}{2}+T+1)} b_1^j b_2^{2k} |a|^{2h-j}. \end{aligned}$$

Let

$$L = \frac{2^{P+A-1}\Gamma(T+1)}{\Gamma(P)\Gamma(A)} \pi^{\frac{N-1}{2}}.$$

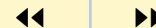
The duplication formula

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$



Title Page

Contents



Page 17 of 40

Go Back

Full Screen

Close

(see [3, p. 45]) is applied with $2z = 2k + P + 2h + A$. Now

$$\Gamma\left(k + h + \frac{A + P + 1}{2}\right) \leq \Gamma\left(k + h + \frac{N}{2} + T + 1\right)$$

since Γ increases on $[1, +\infty[$ and

$$1 \leq k + h + \frac{A + P + 1}{2} \leq k + h + \frac{N}{2} + T + 1.$$

This leads to:

$$\begin{aligned} & I_{A,P,T}(a, b) \\ & \leq L \sum_{(k,h) \in \mathbb{N}^2} \sum_{j=0}^{2h} \frac{(2h)! B(j + 2k + P, 2h - j + A) \Gamma\left(k + h + \frac{A+P}{2}\right)}{h! k! j! (2h - j)!} b_1^j b_2^{2k} |a|^{2h-j} \\ & = L \sum_{(k,h) \in \mathbb{N}^2} \frac{\Gamma\left(k + h + \frac{A+P}{2}\right)}{h! k!} b_2^{2k} \sum_{j=0}^{2h} \frac{(2h)!}{j! (2h - j)!} b_1^j |a|^{2h-j} B(j + 2k + P, 2h - j + A). \end{aligned}$$

The last sum turns into:

$$\begin{aligned} & \sum_{j=0}^{2h} \frac{(2h)! b_1^j |a|^{2h-j}}{j! (2h - j)!} \int_0^1 \xi^{j+2k+P-1} (1 - \xi)^{2h-j+A-1} d\xi \\ & = \int_0^1 \left(\sum_{j=0}^{2h} \frac{(2h)! (b_1 \xi)^j [(1 - \xi) |a|]^{2h-j}}{j! (2h - j)!} \right) \xi^{2k+P-1} (1 - \xi)^{A-1} d\xi \\ & = \int_0^1 [b_1 \xi + |a| (1 - \xi)]^{2h} \xi^{2k+P-1} (1 - \xi)^{A-1} d\xi. \end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 18 of 40

Go Back

Full Screen

Close

Hence the majorant of $I_{A,P,T}(a, b)$ becomes:

$$L \int_0^1 \sum_{k \in \mathbb{N}} \frac{(b_2 \xi)^{2k}}{k!} \left(\sum_{h \in \mathbb{N}} \frac{\Gamma(h + k + \frac{A+P}{2})}{h!} [b_1 \xi + |a| (1 - \xi)]^{2h} \right) \xi^{P-1} (1-\xi)^{A-1} d\xi$$

$$= L \int_0^1 \sum_{k \in \mathbb{N}} \frac{\Gamma(k + \frac{A+P}{2}) (b_2 \xi)^{2k}}{k!} \left(\frac{1}{1 - [b_1 \xi + |a| (1 - \xi)]^2} \right)^{k + \frac{A+P}{2}} \xi^{P-1} (1-\xi)^{A-1} d\xi$$

according to the expansion

$$\frac{\Gamma(C)}{(1 - X)^C} = \sum_{h \in \mathbb{N}} \frac{\Gamma(h + C)}{h!} X^h$$

with $|X| < 1$ when $C > 0$ (see [8, p. 53]). Here $X = [b_1 \xi + |a| (1 - \xi)]^2$ belongs to $] - 1, 1[$ since b_1 and $|a|$ do and $\xi \in [0, 1]$. The same expansion now applies with

$$C = \frac{A + P}{2} \quad \text{and} \quad X = \frac{(b_2 \xi)^2}{1 - [b_1 \xi + |a| (1 - \xi)]^2}$$

since $|X| < 1$, as

$$\begin{aligned} \delta(\xi) &:= (b_2 \xi)^2 + [b_1 \xi + |a| (1 - \xi)]^2 \\ &= |b|^2 \xi^2 + |a|^2 (1 - \xi)^2 + 2\xi(1 - \xi) b_1 |a| \\ &\leq |b|^2 \xi^2 + |a|^2 (1 - \xi)^2 + 2\xi(1 - \xi) |b| |a| \\ &= [\xi |b| + |a| (1 - \xi)]^2 < 1. \end{aligned}$$



Title Page

Contents



Page 19 of 40

Go Back

Full Screen

Close

Hence

$$\begin{aligned}
 & I_{A,P,T}(a, b) \\
 & \leq L \int_0^1 \frac{\Gamma\left(\frac{A+P}{2}\right) \xi^{P-1} (1-\xi)^{A-1} d\xi}{\left(1 - \frac{(b_2 \xi)^2}{1 - [b_1 \xi + |a|(1-\xi)]^2}\right)^{\frac{A+P}{2}} (1 - [b_1 \xi + |a|(1-\xi)]^2)^{\frac{A+P}{2}}} \\
 & = L \cdot \Gamma\left(\frac{A+P}{2}\right) \int_0^1 \frac{\xi^{P-1} (1-\xi)^{A-1} d\xi}{(1 - [b_1 \xi + |a|(1-\xi)]^2 - (b_2 \xi)^2)^{\frac{A+P}{2}}}.
 \end{aligned}$$

Now

$$\begin{aligned}
 1 - \delta(\xi) & \geq 1 - [\xi |b| + |a|(1-\xi)]^2 \\
 & \geq 1 - [\xi |b| + (1-\xi)]^2 \\
 & = \xi(1 - |b|)[2 - \xi(1 - |b|)] \\
 & \geq \xi(1 - |b|^2)
 \end{aligned}$$

since

$$[2 - \xi(1 - |b|)] - (1 + |b|) = (1 - \xi)(1 - |b|) \geq 0.$$

Similarly,

$$1 - \delta(\xi) \geq (1 - \xi)(1 - |a|^2).$$

Thus

$$\frac{1}{[1 - \delta(\xi)]^{\frac{A+P}{2}}} \leq \frac{1}{[(1 - \xi)(1 - |a|^2)]^{\frac{A+P}{2} - \tau} [\xi(1 - |b|^2)]^\tau}$$

since $\tau \geq 0$ and $\frac{A+P}{2} - \tau \geq 0$. Finally:

$$I_{A,P,T}(a, b) \leq \frac{L \cdot \Gamma\left(\frac{A+P}{2}\right)}{(1 - |a|^2)^{\frac{A+P}{2} - \tau} (1 - |b|^2)^\tau} \int_0^1 \xi^{P-\tau-1} (1-\xi)^{A+\tau-\frac{A+P}{2}-1} d\xi.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 20 of 40

Go Back

Full Screen

Close

This integral converges since $P - \tau > 0$ and

$$A + \tau - \frac{A + P}{2} = \frac{A - P}{2} + \tau > 0.$$

Now the result follows with

$$K = L \cdot \Gamma\left(\frac{A + P}{2}\right) B\left(P - \tau, \frac{A - P}{2} + \tau\right) = L \Gamma(P - \tau) \Gamma\left(\frac{A - P}{2} + \tau\right).$$

Proof of Theorem 4.1 in the case $N = 3$. Here

$$I_{A,P,T}(a, b) = \int_0^\pi \int_0^1 \frac{(1 - r^2)^T r^2 (\sin \theta_1) J_P(s, t, b_1, b_2)}{(1 - |a|s)^A} dr d\theta_1,$$

where

$$J_P(s, t, b_1, b_2) = \int_0^{2\pi} \frac{d\theta_2}{(1 - s b_1 - t b_2 \cos \theta_2)^P} = 2 I_P(s, t, b_1, b_2)$$

with $I_P(s, t, b_1, b_2)$ as in Proposition 3.6, with $N = 3$. Hence $I_{A,P,T}(a, b)$ has the same expression as in Formula (4.1), with $N = 3$, since $\sigma_1 = 2$. Thus the proof ends in the same manner as that above in the case $N > 3$.

Proof of Theorem 4.1 in the case $N = 2$. Now $x_1 = s = r \cos \theta$ and $x_2 = t = r \sin \theta$:

$$I_{A,P,T}(a, b) = \int_0^{2\pi} \int_0^1 \frac{(1 - r^2)^T r dr d\theta}{(1 - |a|s)^A (1 - s b_1 - t b_2)^P}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 21 of 40

Go Back

Full Screen

Close

$$\begin{aligned}
 &= \int_{B_2} \sum_{\ell \in \mathbb{N}} \frac{\Gamma(\ell + A)}{\ell! \Gamma(A)} (|a|s)^\ell \sum_{n \in \mathbb{N}} \frac{(tb_2)^n}{n! \Gamma(P)} \frac{\Gamma(n + P)}{(1 - sb_1)^{n+P}} (1 - s^2 - t^2)^T ds dt \\
 &= \sum_{(\ell, n, j) \in \mathbb{N}^3} \frac{\Gamma(\ell + A) |a|^\ell (b_2)^n \Gamma(j + n + P) (b_1)^j}{\ell! \Gamma(A) n! \Gamma(P) j!} \int_{B_2} s^{\ell+j} t^n (1 - s^2 - t^2)^T ds dt.
 \end{aligned}$$

The last integral vanishes when n is odd or $\ell + j$ odd. Otherwise ($n = 2k$ and $\ell + j = 2h$), it equals

$$2 \int_H s^{\ell+j} t^n (1 - s^2 - t^2)^T ds dt = \frac{\Gamma(h + \frac{1}{2}) \Gamma(k + \frac{1}{2}) \Gamma(T + 1)}{\Gamma(k + h + T + 2)}$$

by Lemma 3.4 and turns into

$$\frac{n! (2h)! \pi \Gamma(T + 1)}{2^{2h+2k} h! k! \Gamma(k + h + T + 2)}$$

according to [3, p. 40]. Thus $I_{A,P,T}(a, b)$ is again recognized as Formula (4.2) now with $N = 2$ and the proof ends as for the case $N > 3$. \square

We now present an example of a family of functions $\{f_a\}_a$ which is uniformly bounded above in $\mathcal{Y}_{\alpha, \beta, \gamma}$:

Corollary 4.2. *Given $\beta > -\frac{N+1}{2}$ ($N \geq 2$) let $\alpha = \frac{N+1}{2} + \beta$ and $\gamma > \max(\alpha, -1 - \beta)$. For any $a \in B_N$ let f_a denote the function defined by: $f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^{2\alpha}}$, $\forall x \in B_N$. Then $f_a \in \mathcal{Y}_{\alpha, \beta, \gamma}$, $\forall a \in B_N$. Moreover, there exists $K > 0$ such that $M_{\mathcal{Y}_{\alpha, \beta, \gamma}}(f_a) \leq K$, $\forall a \in B_N$.*

Remark 4. This constant K is the same as that in the previous theorem, with $A = 2\alpha$, $P = 2\gamma$ and $T = \beta + \gamma$.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 22 of 40

Go Back

Full Screen

Close

Proof. With the above choices for parameters A, P, T , we actually have: $P > A > 0, T > -1$ and

$$A + P = 2\alpha + 2\gamma = N + 1 + 2\beta + 2\gamma = N + 1 + 2T > 1.$$

The conditions $0 \leq \tau \leq \alpha + \gamma$ together with $\gamma - \alpha < \tau < 2\gamma$ reduce to: $\gamma - \alpha < \tau \leq \alpha + \gamma$. Let

$$(4.3) \quad J_b(f_a) = (1 - |b|^2)^\alpha \int_{B_N} (1 - |x|^2)^\beta f_a(x) (1 - |\Phi_b(x)|^2)^\gamma dx.$$

Now

$$J_b(f_a) = (1 - |b|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1 - |x|^2)^{\beta+\gamma}}{(1 - \langle x, a \rangle)^{N+1+2\beta} (1 - \langle x, b \rangle)^{2\gamma}} dx$$

$$\leq K \quad \forall a \in B_N, \forall b \in B_N$$

according to Theorem 4.1 applied with $\tau = \alpha + \gamma = \frac{A+P}{2}$. □

4.1. Proof of Theorem 2.2 in the case (i)

Given $R \in]0, 1[$, the subharmonicity of g provides for any $a \in B_N$ the majoration:

$$g(a) \leq \frac{1}{V_a} \int_{B(a, R_a)} g(x) dx$$

with V_a the volume of $B(a, R_a)$. From Lemma 3.3, it is clear that:

$$1 \leq \left(2 \frac{1+R}{1-R} \frac{1-|x|^2}{1-\langle x, a \rangle} \right)^A \quad \forall x \in B(a, R_a)$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 23 of 40

Go Back

Full Screen

Close

with $A = 2\alpha > 0$. Now $g(x) \geq 0, \forall x \in B_N$. With f_a as in Corollary 4.2, this leads to:

$$V_a g(a) \leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} (1-|x|^2)^A f_a(x) g(x) dx.$$

Now

$$A = \alpha + \beta + \frac{N+1}{2} = \alpha + \beta + N - \frac{N-1}{2},$$

thus

$$\begin{aligned} V_a g(a) &\leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\alpha+\beta+N} f_a(x) g(x)}{(1-|x|^2)^{\lambda+\frac{N-1}{2}}} dx \\ &\leq C'K \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{dx}{(1-|x|^2)^{\lambda+\frac{N-1}{2}}} \end{aligned}$$

from Corollary 4.2. Lemmas 1 and 5 of [5] provide

$$\left(\frac{1-|x|^2}{1-|a|^2}\right)^{\lambda+\frac{N-1}{2}} \geq C_{\lambda+\frac{N-1}{2}} \quad \forall x \in B(a, R_a),$$

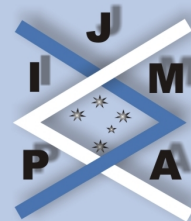
with $C_{\lambda+\frac{N-1}{2}}$ defined in the same pattern as C_β in the proof of Proposition 3.1.

Finally:

$$V_a g(a) \leq \frac{C'K}{C_{\lambda+\frac{N-1}{2}}} \left(2 \frac{1+R}{1-R}\right)^A \frac{V_a}{(1-|a|^2)^{\lambda+\frac{N-1}{2}}},$$

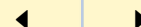
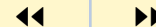
thus

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_{\lambda+\frac{N-1}{2}}} \left(2 \frac{1+R}{1-R}\right)^{2\alpha} \quad \forall R \in]0, 1[.$$



Title Page

Contents



Page 24 of 40

Go Back

Full Screen

Close

The majorant is an increasing function with respect to R . Letting R tend toward 0^+ , we get:

$$M_{X_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_{\lambda+\frac{N-1}{2}}} 2^{2\alpha}.$$

4.2. Proof of Theorem 2.2 in the case (ii)

Here we work with f_a defined by:

$$f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^A} \quad \text{where} \quad A = \alpha + \beta + N.$$

Theorem 4.1 applies once again, with $A = N + 1 + 2\beta > \frac{N-1}{2} > 0$, $P = 2\gamma > 0$ and $T = \beta + \gamma > -1$ (because $\gamma > -1 - \beta$). Condition $A + P = N + 1 + 2T$ is fulfilled too. Moreover $\tau := \alpha + \gamma = \beta + \gamma + 1$ satisfies both $0 \leq \tau \leq \beta + \gamma + \frac{N+1}{2}$ (obviously $0 < \beta + \gamma + 1$ and $1 < \frac{N+1}{2}$) and $\gamma - \beta - \frac{N+1}{2} < \tau < 2\gamma$:

$$\tau - \gamma + \beta + \frac{N+1}{2} = 2\beta + \frac{N+3}{2} > 0 \quad \text{and} \quad 2\gamma - \tau = \gamma - 1 - \beta > 0.$$

With such a choice for τ we have

$$\frac{A+P}{2} - \tau = \frac{N+1}{2} - 1 = \frac{N-1}{2},$$

thus

$$(4.4) \quad I_{A,P,T}(a, b) \leq \frac{K}{(1 - |a|^2)^{\frac{N+1}{2}-1} (1 - |b|^2)^{\alpha+\gamma}} \quad \forall a \in B_N, \forall b \in B_N.$$

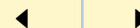
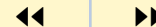
Hence, $J_b(f_a)$ defined in Formula (4.3) now satisfies

$$(4.5) \quad J_b(f_a) \leq \frac{K}{(1 - |a|^2)^{\frac{N-1}{2}}} \quad \forall a \in B_N, \forall b \in B_N.$$



Title Page

Contents



Page 25 of 40

Go Back

Full Screen

Close

In other words,

$$(4.6) \quad M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a) \leq \frac{K}{(1 - |a|^2)^{\frac{N-1}{2}}} \quad \forall a \in B_N.$$

This implies:

$$(4.7) \quad M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(g f_a) \leq \frac{C'K}{(1 - |a|^2)^{\frac{N-1}{2}}} \quad \forall a \in B_N.$$

With R and V_a as in the previous proof, we obtain here:

$$\begin{aligned} V_a g(a) &\leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1 - |x|^2)^{\lambda+\alpha+\beta+N} f_a(x) g(x)}{(1 - |x|^2)^\lambda} dx \\ &\leq \frac{C'K}{(1 - |a|^2)^{\frac{N-1}{2}}} \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{dx}{(1 - |x|^2)^\lambda} \end{aligned}$$

and the last integral is majorized by $\frac{V_a}{C_\lambda (1 - |a|^2)^\lambda}$ with C_λ defined similarly to C_β in the proof of Proposition 3.1. Finally:

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_\lambda} 2^{N+1+2\beta}.$$

4.3. Proof of Theorem 2.2 in the case (iii)

Here f_a is defined by:

$$f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^A (1 - |x|^2)^{\beta+\gamma}} \quad \forall x \in B_N,$$



Title Page

Contents



Page 26 of 40

Go Back

Full Screen

Close

where $A = N + 1 - 2\gamma > 0$. Theorem 4.1 is applied with $P = 2\gamma > 0$ and $T = 0 > -1$. Thus

$$A + P = N + 1 = N + 1 + 2T.$$

We have to choose τ satisfying both

$$0 \leq \tau \leq \frac{N+1}{2} \quad \text{and} \quad 2\gamma - \frac{N+1}{2} < \tau < 2\gamma.$$

Now

$$\tau := \frac{N+1}{2} = \frac{A+P}{2} = \alpha + \gamma$$

fulfills the last condition since:

$$2\gamma - \tau = 2\left(\gamma - \frac{N+1}{4}\right) > 0 \quad \text{and} \quad \tau - 2\gamma + \frac{N+1}{2} = 2\left(\frac{N+1}{2} - \gamma\right) > 0.$$

Formula (4.3) implies $J_b(f_a) \leq K$ for all $a \in B_N$ and all $b \in B_N$. Thus $M_{\mathcal{J}_{\alpha,\beta,\gamma}}(f_a) \leq K, \forall a \in B_N$. As before,

$$V_a g(a) \leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1-|x|^2)^{A+\beta+\gamma} g(x)}{(1-\langle x, a \rangle)^A (1-|x|^2)^{\beta+\gamma}} dx.$$

Now

$$\begin{aligned} A + \beta + \gamma &= N + 1 - \gamma + \beta \\ &= N + 1 + \alpha - \frac{N+1}{2} + \beta \\ &= \alpha + \beta + N - \frac{N-1}{2}, \end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 27 of 40

Go Back

Full Screen

Close

whence

$$\begin{aligned} V_a g(a) &\leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a, R_a)} \frac{(1-|x|^2)^{\alpha+\beta+N} f_a(x) g(x)}{(1-|x|^2)^{\frac{N-1}{2}}} dx \\ &\leq C' K \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a, R_a)} \frac{dx}{(1-|x|^2)^{\lambda+\frac{N-1}{2}}} \end{aligned}$$

and the proof ends as in the case (i). Here

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C' K}{C_{\lambda+\frac{N-1}{2}}} 2^{N+1-2\gamma}.$$

4.4. Proof of Theorem 2.2 in the case (iv)

Here f_a is defined by:

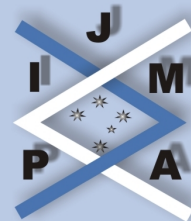
$$f_a(x) = \frac{1}{(1-\langle x, a \rangle)^A (1-|x|^2)^\beta} \quad \forall x \in B_N,$$

where $A = N + 1$, $T = \gamma$ and $P = 2\gamma$ thus $A + P = N + 1 + 2T$, allowing us to use Theorem 4.1, with $\tau = \alpha + \gamma = 1 + \gamma$ (since $0 \leq \tau \leq \frac{N+1}{2} + \gamma$ and $\gamma - \frac{N+1}{2} < \tau < 2\gamma$). Hence Inequalities (4.4), (4.5), (4.6) and (4.7) follow. Now

$$(4.8) \quad V_a g(a) \leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a, R_a)} (1-|x|^2)^{A+\beta} f_a(x) g(x) dx.$$

Since $A + \beta = \alpha + \beta + N$, this turns into:

$$V_a g(a) \leq \left(2 \frac{1+R}{1-R}\right)^A \int_{B(a, R_a)} \frac{M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(g f_a)}{(1-|x|^2)^\lambda} dx$$



and the proof ends as in the case (ii), here with:

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_\lambda} 2^{N+1}.$$

4.5. Proof of Theorem 2.2 in the case (v)

Here f_a is defined by:

$$f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^A (1 - |x|^2)^\gamma} \quad \forall x \in B_N,$$

where

$$A = N + 1 + 2(\beta - \gamma) > N + 1 - \frac{N + 3}{2} = \frac{N - 1}{2} > 0.$$

With $P = 2\gamma > 0$ and $T = \beta$, the condition $A + P = N + 1 + 2T$ of Theorem 4.1 is fulfilled. Moreover $\tau := \alpha + \gamma = 1 + \beta$ satisfies

$$0 \leq \tau \leq \frac{N + 1}{2} + \beta \quad \text{and} \quad 2\gamma - \frac{N + 1}{2} - \beta < \tau < 2\gamma$$

since:

$$2\gamma - \tau = 2\gamma - (1 + \beta) > 0 \quad \text{and} \quad \tau - 2\gamma + \frac{N + 1}{2} + \beta = -2\gamma + \frac{N + 3}{2} + 2\beta > 0.$$

Again

$$\frac{A + P}{2} - \tau = \frac{N + 1}{2} - 1 = \frac{N - 1}{2}$$

and inequalities (4.4) to (4.7) follow. Formula (4.8) still holds with $(1 - |x|^2)^{A+\gamma}$ instead of $(1 - |x|^2)^{A+\beta}$. Here

$$A + \gamma = N + 1 + 2\beta - \gamma = N + \alpha + \beta$$



Title Page

Contents



Page 29 of 40

Go Back

Full Screen

Close

and the conclusion follows as in the previous case. Finally:

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_\lambda} 2^{N+1+2(\beta-\gamma)}.$$

4.6. Proof of Theorem 2.2 in the case (vi)

Here f_a is defined by:

$$f_a(x) = \frac{1}{(1 - \langle x, a \rangle)^A (1 - |x|^2)^\alpha} \quad \forall x \in B_N$$

with $A = N + \beta > \frac{N-1}{2} > 0$, $P = 2\gamma > 0$, $T = \frac{\beta-1}{2} + \gamma > -1$ (actually $T + 1 = \frac{\beta+1}{2} + \gamma > 0$). The use of Theorem 4.1 is allowed since

$$A + P = N + 1 + \beta - 1 + 2\gamma = N + 1 + 2T.$$

Now $\tau := \alpha + \gamma = \frac{\beta+1}{2} + \gamma$ satisfies $0 \leq \tau \leq \frac{N+\beta}{2} + \gamma$ (because of $\gamma > -\frac{\beta+1}{2}$). Moreover $\gamma - \frac{N+\beta}{2} < \tau < 2\gamma$ is fulfilled too since

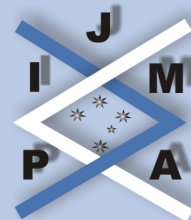
$$\frac{\beta+1}{2} < \gamma \quad \text{and} \quad \beta + 1 + (N + \beta) = 1 + N + 2\beta > 0.$$

In addition,

$$\frac{A + P}{2} - \tau = \frac{N + \beta}{2} - \frac{\beta + 1}{2} = \frac{N - 1}{2}.$$

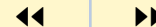
Again it induces Formula (4.6). With $(1 - |x|^2)^{A+\beta}$ replaced by $(1 - |x|^2)^{A+\alpha}$, inequality (4.8) remains valid. Since $A + \alpha = N + \alpha + \beta$, the conclusion is once again obtained in a similar way as in the cases (iv) and (v), here with

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_\lambda} 2^{N+\beta}.$$



Title Page

Contents



Page 30 of 40

Go Back

Full Screen

Close

5. The Situation with Radial Subharmonic Functions

5.1. The example of $u : x \mapsto (1 - |x|^2)^{-A}$ with $A \geq 0$

Proposition 5.1. Given $P \geq 1$, $T > -1$ and $N \in \mathbb{N}$ ($N \geq 2$) such that $P \leq N + 1 + 2T$, let

$$I_{P,T}(b) = \int_{B_N} \frac{(1 - |x|^2)^T}{(1 - \langle x, b \rangle)^P} dx \quad \forall b \in B_N.$$

Then

$$I_{P,T}(b) \leq \frac{K'}{(1 - |b|^2)^{P/2}} \quad \forall b \in B_N,$$

(equality holds when $P = N + 1 + 2T$) with

$$K' = \frac{\Gamma(T + 1)}{\Gamma\left(\frac{P+1}{2}\right)} \pi^{\frac{N}{2}}.$$

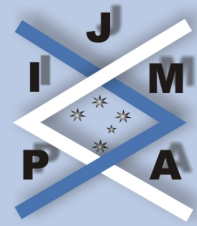
Proof. Letting $A \rightarrow 0^+$ in Theorem 4.1, the majorization of $I_{P,T}(b)$ is an immediate result, since K (as a function of A) tends towards K' : see Example 4.1. Nonetheless, we still have to show that equality holds in the case $P = N + 1 + 2T$.

Proof in the case $N \geq 3$. Up to a unitary transform, we may assume $b = (|b|, 0, 0, \dots, 0)$, so that $\langle x, b \rangle = |b| x_1 = |b| r \cos \theta_1$ with $\theta_1 \in]0, \pi[$ (we will have $\theta_1 \in]0, 2\pi[$ in the case $N = 2$). Now

$$dx = r^{N-1} (\sin \theta_1)^{N-2} dr d\theta_1 d\sigma^{(N-1)},$$

with the same notations as in the proof of Theorem 4.1. Here:

$$I_{P,T}(b) = \sigma_{N-1} \int_0^\pi \int_0^1 \frac{(1 - r^2)^T r^{N-1} (\sin \theta_1)^{N-2}}{(1 - |b| r \cos \theta_1)^P} dr d\theta_1.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 31 of 40

Go Back

Full Screen

Close

Then

$$(5.1) \quad I_{P,T}(b) = \sigma_{N-1} \sum_{n \in \mathbb{N}} \frac{\Gamma(n+P)}{n! \Gamma(P)} |b|^n \iint_H s^n t^{N-2} (1-s^2-t^2)^T ds dt$$

with $s = r \cos \theta_1$ and $t = r \sin \theta_1$. This integral vanishes for odd n . If $n = 2k$, its value is given by Lemma 3.4. Thus

$$I_{P,T}(b) = \frac{\sigma_{N-1} \Gamma\left(\frac{N-1}{2}\right) \Gamma(T+1)}{2 \Gamma(P)} \sum_{k \in \mathbb{N}} \frac{|b|^{2k} \Gamma\left(k + \frac{1}{2}\right) \Gamma(2k+P)}{(2k)! \Gamma\left(k + \frac{N}{2} + T + 1\right)}.$$

Now [2, p. 29] and [3, p. 40] lead to:

$$I_{P,T}(b) = \frac{\Gamma(T+1)}{\Gamma(P)} \pi^{\frac{N-1}{2}} \sum_{k \in \mathbb{N}} \frac{|b|^{2k} \sqrt{\pi} \Gamma(2k+P)}{2^{2k} k! \Gamma\left(k + \frac{N}{2} + T + 1\right)}.$$

Through the duplication formula ([3, p. 45]), it follows that:

$$\begin{aligned} I_{P,T}(b) &= \frac{\Gamma(T+1)}{\Gamma(P)} \pi^{\frac{N-1}{2}} \sum_{k \in \mathbb{N}} \frac{|b|^{2k} 2^{2k+P-1} \Gamma\left(k + \frac{P}{2}\right) \Gamma\left(k + \frac{P+1}{2}\right)}{2^{2k} k! \Gamma\left(k + \frac{N}{2} + T + 1\right)} \\ &= K' \sum_{k \in \mathbb{N}} \frac{\Gamma\left(k + \frac{P}{2}\right)}{k! \Gamma\left(\frac{P}{2}\right)} |b|^{2k} \end{aligned}$$

with

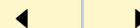
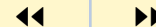
$$K' = \frac{\Gamma(T+1)}{\Gamma(P)} \pi^{\frac{N-1}{2}} 2^{P-1} \Gamma\left(\frac{P}{2}\right).$$

Another application of the duplication formula provides the final expression of K' .



Title Page

Contents



Page 32 of 40

Go Back

Full Screen

Close

Proof in the case $N = 2$. Now

$$I_{P,T}(b) = \int_0^{2\pi} \int_0^1 \frac{(1-r^2)^T r}{(1-|b|r \cos \theta)^P} dr d\theta.$$

Then

$$I_{P,T}(b) = \sum_{n \in \mathbb{N}} \frac{\Gamma(n+P)}{n! \Gamma(P)} |b|^n \left(\int_0^1 r^{n+1} (1-r^2)^T dr \right) \left(\int_0^{2\pi} (\cos \theta)^n d\theta \right).$$

The last integral equals $2 \int_0^\pi (\cos \theta)^n d\theta$ for any n . As $\sigma_1 = 2$, here we recognize the same expression as in formula (5.1), replacing N by 2. Hence the same conclusion. \square

Corollary 5.2. *Given $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$ and $\gamma > \frac{N-1}{2}$, let $A = \frac{N+1}{2} + \beta$ and u defined on B_N by:*

$$u(x) = \frac{1}{(1-|x|^2)^A} \quad \forall x \in B_N.$$

Then $u \in \mathcal{RSY}_{\alpha,\beta,\gamma}$ and $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \leq K'$ where K' stems from Proposition 5.1 (with $P = 2\gamma > 1$ and $T = \beta + \gamma - A = \gamma - \frac{N+1}{2} > -1$).

Proof. The subharmonicity of u follows from Lemma 3.5 since $A \geq 0$. Let $J_b(u)$ be defined similarly as in formula (4.3). Then

$$J_b(u) = (1-|b|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1-|x|^2)^{\beta+\gamma-A}}{(1-\langle x, b \rangle)^P} dx.$$

As

$$N+1+2T = N+1+2\gamma - (N+1) = P,$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 33 of 40

Go Back

Full Screen

Close

Proposition 5.1 provides:

$$J_b(u) \leq (1 - |b|^2)^{\alpha+\gamma} \frac{K'}{(1 - |b|^2)^{P/2}} \leq K'$$

since $\alpha \geq 0$. The conclusion proceeds from

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) = \sup_{b \in B_N} J_b(u).$$

□

5.2. Proof of Theorem 2.3

Let A and u be defined as in Corollary 5.2. With R and V_a as in the proof of Theorem 2.2:

$$\begin{aligned} V_a g(a) &\leq \int_{B(a,R_a)} (1 - |x|^2)^A u(x) g(x) dx \\ &= \int_{B(a,R_a)} \frac{(1 - |x|^2)^{\lambda+\alpha+\beta+N} u(x) g(x) dx}{(1 - |x|^2)^{\lambda+\alpha+\frac{N-1}{2}}} \end{aligned}$$

since:

$$A = \frac{N+1}{2} + \beta = \beta + N - \frac{N-1}{2}.$$

This leads to:

$$\begin{aligned} V_a g(a) &\leq C''' K' \int_{B(a,R_a)} \frac{dx}{(1 - |x|^2)^{\lambda+\alpha+\frac{N-1}{2}}} \\ &\leq \frac{C''' K' V_a}{C_{\lambda+\alpha+\frac{N-1}{2}}} \frac{1}{(1 - |a|^2)^{\lambda+\alpha+\frac{N-1}{2}}}, \end{aligned}$$

with $C_{\lambda+\alpha+\frac{N-1}{2}}$ defined in the same way as C_β in the proof of Proposition 3.1. We obtain finally:

$$M_{\mathcal{X}_{\lambda+\alpha+\frac{N-1}{2}}}(g) \leq \frac{C'' K'}{C_{\lambda+\alpha+\frac{N-1}{2}}}.$$



**Multiplication of
Subharmonic Functions**

R. Supper

vol. 9, iss. 4, art. 118, 2008

Title Page

Contents



Page 34 of 40

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 35 of 40

Go Back

Full Screen

Close

6. Annex: The Sets $\mathcal{S}\mathcal{X}_\lambda$ and $\mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma}$ for some Special Values of $\lambda, \alpha, \beta, \gamma$

Throughout the paper, it was assumed that $\gamma \geq 0$. When $\gamma \leq 0$, the set $\mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma}$ is related to other sets of the same kind by:

Proposition 6.1. *Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma \leq 0$, then*

$$\mathcal{Y}_{\alpha+\gamma,\beta+\gamma,0}^+ \subset \mathcal{Y}_{\alpha,\beta,\gamma}^+ \subset \mathcal{Y}_{\alpha+s\gamma,\beta-s\gamma,0}^+ \quad \forall s \in [-1, 1],$$

where $\mathcal{Y}_{\alpha,\beta,\gamma}^+$ denotes the subset of $\mathcal{Y}_{\alpha,\beta,\gamma}$ consisting of all non-negative $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ (not necessarily subharmonic).

Proof. For any $a \in B_N$ and $x \in B_N$, the following holds:

$$(6.1) \quad (1 - |a|^2)^\alpha (1 - |x|^2)^\beta (1 - |\Phi_a(x)|^2)^\gamma \\ = (1 - |a|^2)^{\alpha+\gamma} (1 - |x|^2)^{\beta+\gamma} (1 - \langle a, x \rangle)^{-2\gamma}.$$

Since $\langle a, x \rangle \in]-1, 1[$ through the Cauchy-Schwarz inequality, we have $(1 - \langle a, x \rangle)^{-2\gamma} \leq 2^{-2\gamma}$ as $-2\gamma \geq 0$. If $u \in \mathcal{Y}_{\alpha+\gamma,\beta+\gamma,0}$ and $u(x) \geq 0, \forall x \in B_N$, then $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ with

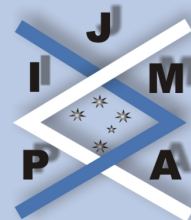
$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \leq 2^{-2\gamma} M_{\mathcal{Y}_{\alpha+\gamma,\beta+\gamma,0}}(u).$$

Also, $\langle a, x \rangle < |a|$ and $\langle a, x \rangle < |x|$, thus

$$(1 - \langle a, x \rangle)^{(s-1)\gamma} \geq (1 - |a|)^{(s-1)\gamma} \quad \text{and} \quad (1 - \langle a, x \rangle)^{(-s-1)\gamma} \geq (1 - |x|)^{(-s-1)\gamma}$$

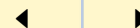
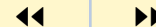
since $(s-1)\gamma \geq 0$ and $(-s-1)\gamma \geq 0$. Moreover

$$1 - |a| = \frac{1 - |a|^2}{1 + |a|} \geq \frac{1 - |a|^2}{2} \quad \text{and} \quad 1 - |x| \geq \frac{1 - |x|^2}{2},$$



Title Page

Contents



Page 36 of 40

Go Back

Full Screen

Close

thus

$$(1 - \langle a, x \rangle)^{-2\gamma} \geq (1 - |a|^2)^{(s-1)\gamma} (1 - |x|^2)^{(-s-1)\gamma} \left(\frac{1}{2}\right)^{-2\gamma}.$$

Finally

$$(1 - |a|^2)^\alpha (1 - |x|^2)^\beta (1 - |\Phi_a(x)|^2)^\gamma \geq 2^{2\gamma} (1 - |a|^2)^{\alpha+s\gamma} (1 - |x|^2)^{\beta-s\gamma}.$$

Any non-negative $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ then belongs to $\mathcal{Y}_{\alpha+s\gamma,\beta-s\gamma,0}$ with

$$M_{\mathcal{Y}_{\alpha+s\gamma,\beta-s\gamma,0}}(u) \leq 2^{-2\gamma} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u).$$

□

Remark 5. Even with $\gamma \leq 0$, Proposition 3.1 still holds, since

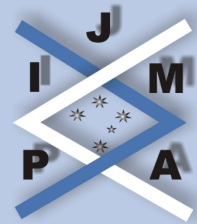
$$\begin{aligned} (1 - |\Phi_a(x)|^2)^\gamma &= \left(\frac{1 - \langle a, x \rangle}{1 - |x|^2}\right)^{-\gamma} \left(\frac{1 - \langle a, x \rangle}{1 - |a|^2}\right)^{-\gamma} \\ &\geq \left(\frac{1}{2}\right)^{-\gamma} \left(\frac{1}{4}\right)^{-\gamma} = 2^{3\gamma} \quad \forall x \in B(a, R_a) \end{aligned}$$

according to Lemma 3.3. For the proof of Proposition 3.1 in the case $\gamma \leq 0$, it is enough to replace $(1 - R^2)^\gamma$ in formula (3.1) by $2^{3\gamma}$.

Proposition 6.2. *If $\lambda < 0$, then the set $\mathcal{S}\mathcal{X}_\lambda$ contains only the function $u \equiv 0$ on B_N .*

Proof. Given $u \in \mathcal{S}\mathcal{X}_\lambda$ and $\xi \in B_N$, let $r \in]|\xi|, 1[$. Then

$$u(\xi) \leq \max_{|x| \leq r} u(x) = \max_{|x|=r} u(x)$$



Title Page

Contents



Page 37 of 40

Go Back

Full Screen

Close

according to the maximum principle (see [2, pp. 48–49]). Thus

$$0 \leq u(\xi) \leq M_{\mathcal{X}_\lambda}(u) (1 - r^2)^{-\lambda}$$

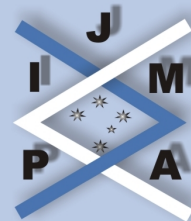
which tends towards 0 as $r \rightarrow 1^-$ (since $-\lambda > 0$). Finally $u(\xi) = 0$. □

Remark 6. When $\alpha < 0$, it is not compulsory that $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$. For instance, with α, β, γ as in case (ii) of Theorem 2.2, we have $\alpha = \beta + 1 > \frac{1-N}{4}$. It is thus possible to choose β in such a way that $\alpha < 0$. In Subsection 4.2 we have an example of function $f_a \in \mathcal{SY}_{\alpha,\beta,\gamma}$ (with a fixed in B_N) and this function is not vanishing. Similarly $\beta < 0$ does not imply $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$. In Table 1 we have several examples of such situations: see Subsections 4.1 to 4.6 for examples of non-vanishing subharmonic functions belonging to such sets $\mathcal{SY}_{\alpha,\beta,\gamma}$.

Proposition 6.3. *Let $\gamma \in \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha + \beta < -N$, then $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$.*

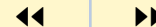
Proof. Given $R \in]0, 1[$, let $K_{R,\gamma} = (1 - R^2)^\gamma$ if $\gamma \geq 0$, or $K_{R,\gamma} = 2^{3\gamma}$ if $\gamma \leq 0$. Then: $(1 - |\Phi_a(x)|^2)^\gamma \geq K_{R,\gamma}, \forall a \in B_N, \forall x \in B(a, R_a)$ according to Remark 5 (also remember that $|\Phi_a| < R$ on $B(a, R_a)$, see [6]). With C_β as in the proof of Proposition 3.1, the following inequalities hold for any $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$ and any $a \in B_N$. The second inequality is based upon $u \geq 0$ and the last one makes use of the suharmonicity of u .

$$\begin{aligned} (1 - |a|^2)^{-\alpha} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) &\geq \int_{B_N} (1 - |x|^2)^\beta u(x) (1 - |\Phi_a(x)|^2)^\gamma dx \\ &\geq K_{R,\gamma} \int_{B(a,R_a)} (1 - |x|^2)^\beta u(x) dx \end{aligned}$$



Title Page

Contents



Page 38 of 40

Go Back

Full Screen

Close

$$\begin{aligned} &\geq K_{R,\gamma} C_\beta (1 - |a|^2)^\beta \int_{B(a,R_a)} u(x) dx \\ &\geq K_{R,\gamma} C_\beta (1 - |a|^2)^\beta V_a u(a) \end{aligned}$$

where the volume V_a of $B(a, R_a)$ satisfies:

$$V_a \geq \frac{\sigma_N}{N} \left(\frac{R}{1+R} \right)^N (1 - |a|^2)^N$$

(see the end of the proof of Proposition 3.1). Thus

$$u(a) \leq \kappa (1 - |a|^2)^{-\alpha-\beta-N} \quad \forall a \in B_N,$$

the constant $\kappa > 0$ being independent of a .

Given $\xi \in B_N$, the maximum principle now provides for any $r \in]|\xi|, 1[$:

$$0 \leq u(\xi) \leq \max_{|x| \leq r} u(x) = \max_{|x|=r} u(x) \leq \kappa (1 - r^2)^{-\alpha-\beta-N}$$

which tends towards 0 as $r \rightarrow 1^-$, since $-\alpha - \beta - N > 0$. Hence $u(\xi) = 0$. \square

Proposition 6.4. *Given $\gamma \geq 0$, $\alpha < -\gamma$ and $\beta \in \mathbb{R}$, then $\mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma} = \{0\}$.*

Proof. Since $1 - \langle x, a \rangle \in]0, 2[$, we have $(1 - \langle x, a \rangle)^{-2\gamma} \geq 2^{-2\gamma}$, $\forall x \in B_N, \forall a \in B_N$. Given $u \in \mathcal{S}\mathcal{Y}_{\alpha,\beta,\gamma}$, $\xi \in B_N$ and $r \in]0, 1 - |\xi|[$, the formula (6.1) leads to:

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq (1 - |a|^2)^{\alpha+\gamma} 2^{-2\gamma} \int_{B(\xi,r)} (1 - |x|^2)^{\beta+\gamma} u(x) dx \quad \forall a \in B_N$$

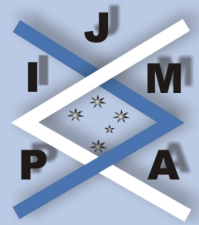
since $u \geq 0$ on $B_N \supset B(\xi, r)$. Now $|x| \leq |\xi| + r$, $\forall x \in B(\xi, r)$. Let $L_\xi = [1 - (|\xi| + r)^2]^{\beta+\gamma}$ if $\beta + \gamma \geq 0$, or $L_\xi = 1$ if $\beta + \gamma \leq 0$. Then

$$(1 - |a|^2)^{-\alpha-\gamma} 2^{2\gamma} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq L_\xi \int_{B(\xi,r)} u(x) dx \geq L_\xi \frac{\sigma_N}{N} r^N u(\xi) \quad \forall a \in B_N$$

since u is subharmonic and the volume of $B(\xi, r)$ is $\frac{\sigma_N}{N} r^N$. Finally, with ξ fixed, we have:

$$0 \leq u(\xi) \leq \kappa_\xi (1 - |a|^2)^{-\alpha-\gamma} \quad \forall a \in B_N,$$

the constant $\kappa_\xi > 0$ being independent of a . Hence $u(\xi) = 0$, letting $|a| \rightarrow 1^-$. \square



**Multiplication of
Subharmonic Functions**

R. Supper

vol. 9, iss. 4, art. 118, 2008

Title Page

Contents



Page 39 of 40

Go Back

Full Screen

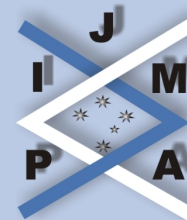
Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

References

- [1] A.B. ALEKSANDROV, Function theory in the ball, in: *Several Complex Variables, Part II*, Encyclopedia of Mathematical Sciences, Volume 8, Springer Verlag, 1994.
- [2] W.K. HAYMAN AND P.B. KENNEDY, *Subharmonic Functions, Vol. I*, London Mathematical Society Monographs, No. 9. Academic Press, London–New York, 1976.
- [3] R. REMMERT, *Classical Topics in Complex Function Theory*, Graduate Texts in Mathematics, 172. Springer-Verlag, New York, 1998.
- [4] W. RUDIN, Function theory in the unit ball of \mathbb{C}^n , *Fundamental Principles of Mathematical Science*, 241, Springer-Verlag, New York-Berlin, 1980.
- [5] R. SUPPER, Bloch and gap subharmonic functions, *Real Analysis Exchange*, **28**(2) (2003), 395–414.
- [6] R. SUPPER, Subharmonic functions with a Bloch type growth, *Integral Transforms and Special Functions*, **16**(7) (2005), 587–596.
- [7] R. YONEDA, Pointwise multipliers from $BMOA^\alpha$ to $BMOA^\beta$, *Complex Variables, Theory and Applications*, **49**(14) (2004), 1045–1061.
- [8] K.H. ZHU, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139. Marcel Dekker, Inc., New York, 1990.
- [9] C. ZUILY, *Distributions et équations aux dérivées partielles*, Collection Méthodes, Hermann, Paris, 1986.



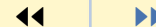
Multiplication of
Subharmonic Functions

R. Supper

vol. 9, iss. 4, art. 118, 2008

Title Page

Contents



Page 40 of 40

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756