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DIRECT RESULTS FOR CERTAIN FAMILY OF INTEGRAL TYPE OPERATORS

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ABSTRACT. In the present paper we introduce a certain family of linear positive operators and study some direct results which include a pointwise convergence, asymptotic formula and an estimation of error in simultaneous approximation.

Key words and phrases: Linear positive operators, Simultaneous approximation, Steklov mean, Modulus of continuity.

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1. INTRODUCTION

We consider a certain family of integral type operators, which are defined as

$$B_n(f,x) = \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^\infty p_{n,\nu-1}(t)f(t)dt + (1+x)^{-n-1}f(0), \quad x\in[0,\infty), \tag{1.1}$$

where

$$p_{n,\nu}(t) = \frac{1}{B(n,\nu+1)}t^{\nu}(1+t)^{-n-\nu-1}$$

with $B(n,\nu+1) = \nu!(n-1)!/(n+\nu)!$ the Beta function.

Alternatively the operators (1.1) may be written as

$$B_n(f,x) = \int_0^\infty W_n(x,t)f(t)dt,$$

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where

$$W_n(x,t) = \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)p_{n,\nu-1}(t) + (1+x)^{-n-1}\delta(t),$$

$\delta(t)$ being the Dirac delta function. The operators B_n are discretely defined linear positive operators. It is easily verified that these operators reproduce only the constant functions. As far as the degree of approximation is concerned these operators are very similar to the operators considered by Srivastava and Gupta [5], but the approximation properties of these operators are different. In this paper, we study some direct theorems in simultaneous approximation for the operators (1.1).

2. AUXILIARY RESULTS

In this section we mention some lemmas which are necessary to prove the main theorems.

Lemma 2.1 ([3]). *For $m \in \mathbb{N}^0$, if the m-th order moment is defined as*

$$U_{n,m}(x) = \frac{1}{n}\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\left(\frac{\nu}{n+1} - x\right)^m,$$

then

$$(n+1)U_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right].$$

Consequently

$$U_{m,n}(x) = \mathcal{O}\left(n^{-[(m+1)/2]}\right).$$

Lemma 2.2. *Let the function $\mu_{n,m}(x)$, $n > m$ and $m \in \mathbb{N}^0$, be defined as*

$$\mu_{n,m}(x) = \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m dt + (-x)^m(1+x)^{-n-1}.$$

Then

$$\mu_{n,0}(x) = 1,\ \mu_{n,1}(x) = \frac{2x}{(n-1)},\ \mu_{n,2}(x) = \frac{2nx(1+x)+2x(1+4x)}{(n-1)(n-2)}$$

and there holds the recurrence relation

$$\begin{aligned}(n-m-1)&\mu_{n,m+1}(x)\\ &= x(1+x)\left[\mu'_{n,m}(x) + 2m\mu_{n,m-1}(x)\right] + \left[m(1+2x)+2x\right]\mu_{n,m}(x).\end{aligned}$$

Consequently for each $x \in [0,\infty)$, we have from this recurrence relation that

$$\mu_{n,m}(x) = \mathcal{O}\left(n^{-[(m+1)/2]}\right).$$

Proof. The values of $\mu_{n,0}(x)$, $\mu_{n,1}(x)$ and $\mu_{n,2}(x)$ easily follow from the definition. We prove the recurrence relation as follows

$$\begin{aligned}x(1+x)\mu'_{n,m}(x) &= \frac{1}{n}\sum_{\nu=1}^{\infty} x(1+x)p'_{n,\nu}(x)\int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m dt\\ &\quad - m\frac{1}{n}\sum_{\nu=1}^{\infty} x(1+x)p_{n,\nu}(x)\int_0^{\infty} p_{n,\nu-1}(t)(t-x)^{m-1}dt\\ &\quad - \left[(n+1)(-x)^m(1+x)^{-n-2} + m(-x)^{m-1}(1+x)^{-n-1}\right]x(1+x).\end{aligned}$$

Now using the identities $x(1+x)p'_{n,\nu}(x) = [\nu - (n+1)x]\, p_{n,\nu}(x)$, we obtain

$$\begin{aligned}
&x(1+x)\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right] \\
&\quad = \frac{1}{n}\sum_{\nu=1}^{\infty} [\nu - (n+1)x]\, p_{n,\nu}(x) \\
&\qquad \times \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m dt + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\
&\quad = \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} [\{(\nu-1)-(n+1)t\} + (n+1)(t-x)+1]\, p_{n,\nu-1}(t)(t-x)^m dt \\
&\qquad + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\
&\quad = \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} t(1+t)p'_{n,\nu-1}(t)(t-x)^m dt \\
&\qquad + (n+1)\frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^{m+1} dt \\
&\qquad + \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m dt + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\
&\quad = \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} \left[(1+2x)(t-x) + (t-x)^2 + x(1+x)\right] p'_{n,\nu-1}(t)(t-x)^m dt \\
&\qquad + (n+1)\frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^{m+1} dt \\
&\qquad + \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x) \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^m dt + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\
&\quad = -(m+1)(1+2x)\left[\mu_{n,m}(x) - (-x)^m(1+x)^{-n-1}\right] \\
&\qquad - (m+2)\left[\mu_{n,m+1}(x) - (-x)^{m+1}(1+x)^{-n-1}\right] \\
&\qquad - mx(1+x)\left[\mu_{n,m-1}(x) - (-x)^{m-1}(1+x)^{-n-1}\right] \\
&\qquad + (n+1)\left[\mu_{n,m+1}(x) - (-x)^{m+1}(1+x)^{-n-1}\right] \\
&\qquad + \left[\mu_{n,m}(x) - (-x)^m(1+x)^{-n-1}\right] + (n+1)(-x)^{m+1}(1+x)^{-n-1} \\
&\quad = -[m(1+2x)+2x]\,\mu_{n,m}(x) + (n-m-1)\mu_{n,m+1}(x) - mx(1+x)\mu_{n,m-1}(x).
\end{aligned}$$

This completes the proof of recurrence relation. □

Remark 2.3. It is easily verified from Lemma 2.2 and by the principle of mathematical induction, that for $n > i$ and each $x \in (0,\infty)$

$$\begin{aligned}
B_n(t^i, x) &= \frac{(n+i)!(n-i-1)!}{n!(n-1)!}x^i \\
&\qquad + i(i-1)\frac{(n+i-1)!(n-i-1)!}{n!(n-1)!}x^{i-1} + i(i-1)(i-2)\mathcal{O}\left(n^{-2}\right).
\end{aligned}$$

Corollary 2.4. *Let δ be a positive number. Then for every $n > \gamma > 0$, $x \in (0,\infty)$, there exists a constant $M(s,x)$ independent of n and depending on s and x such that*

$$\frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_{|t-x|>\delta} p_{n,\nu-1}(t)t^{\gamma}dt \leq M(s,x)n^{-s}, \quad s = 1,2,3,\ldots.$$

Lemma 2.5 ([3])**.** *There exist the polynomials $Q_{i,j,r}(x)$ independent of n and ν such that*

$$\{x(1+x)\}^r D^r[p_{n,\nu}(x)] = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+1)^i[\nu-(n+1)x]^j Q_{i,j,r}(x)p_{n,\nu}(x),$$

where $D \equiv \frac{d}{dx}$.

Lemma 2.6. *Let f be r times differentiable on $[0,\infty)$ such that $f^{(r-1)}$ is absolutely continuous with $f^{(r-1)}(t) = \mathcal{O}(t^{\gamma})$ for some $\gamma > 0$ as $t \to \infty$. Then for $r = 1,2,3,\ldots$ and $n > \gamma + r$ we have*

$$B_n^{(r)}(f,x) = \frac{(n+r-1)!(n-r-1)!}{n!(n-1)!}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} p_{n-r,\nu+r-1}(t)f^{(r)}(t)dt.$$

Proof. It follows by simple computation the following relations:

$$p'_{n,\nu}(t) = n\left[p_{n+1,\nu-1}(t) - p_{n+1,\nu}(t)\right], \tag{2.1}$$

where $t \in [0,\infty)$.

Furthermore, we prove our lemma by mathematical induction. Using the above identity (2.1), we have

$$\begin{aligned} B'_n(f,x) &= \frac{1}{n}\sum_{\nu=1}^{\infty} p'_{n,\nu}(x)\int_0^{\infty} p_{n,\nu-1}(t)f(t)dt - (n+1)(1+x)^{-n-2}f(0) \\ &= \sum_{\nu=1}^{\infty}\left[p_{n+1,\nu-1}(x) - p_{n+1,\nu}(x)\right]\int_0^{\infty} p_{n,\nu-1}(t)f(t)dt \\ &\qquad - (n+1)(1+x)^{-n-2}f(0) \\ &= np_{n+1,0}(x)\int_0^{\infty} p_{n,0}(t)f(t)dt - (n+1)(1+x)^{-n-2}f(0) \\ &\qquad + \sum_{\nu=1}^{\infty} p_{n+1,\nu}(x)\int_0^{\infty}\left[p_{n,\nu}(t) - p_{n,\nu-1}(t)\right]f(t)dt \\ &= (n+1)(1+x)^{-n-2}\int_0^{\infty} n(1+t)^{-n-1}f(t)dt \\ &\qquad + \sum_{\nu=1}^{\infty} p_{n+1,\nu}(x)\int_0^{\infty}\left(\frac{-1}{n-1}\right)p'_{n-1,\nu}(t)f(t)dt \\ &\qquad - (n+1)(1+x)^{-n-2}f(0)\,. \end{aligned}$$

Applying the integration by parts, we get

$$\begin{aligned} B_n'(f,x) &= (n+1)(1+x)^{-n-2}f(0) + (n+1)(1+x)^{-n-2}\int_0^\infty (1+t)^{-n}f'(t)dt \\ &\quad + \frac{1}{n-1}\sum_{\nu=1}^{\infty} p_{n+1,\nu}(x)\int_0^\infty p_{n-1,\nu}(t)f'(t)dt - (n+1)(1+x)^{-n-2}f(0) \\ &= \frac{1}{n-1}\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x)\int_0^\infty p_{n-1,\nu}(t)f'(t)dt\ , \end{aligned}$$

which was to be proved.

If we suppose that

$$B_n^{(i)}(f,x) = \frac{(n+i-1)!(n-i-1)!}{n!(n-1)!}\sum_{\nu=0}^{\infty} p_{n+i,\nu}(x)\int_0^\infty p_{n-i,\nu+i-1}(t)f^{(i)}(t)dt$$

then by (2.1), and using a similar method to the one above it is easily verified that the result is true for $r = i+1$. Therefore by the principle of mathematical induction the result follows. □

3. SIMULTANEOUS APPROXIMATION

In this section we study the rate of pointwise convergence of an asymptotic formula and an error estimation in terms of a higher order modulus of continuity in simultaneous approximation for the operators defined by (1.1). Throughout the section, we have $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \le Mt^\gamma \text{ for some } M>0,\ \gamma>0\}$.

Theorem 3.1. *Let $f \in C_\gamma[0,\infty)$, $\gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then*

$$B_n^{(r)}(f,x) = f^{(r)}(x) + o(1) \text{ as } n \to \infty.$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

Hence

$$\begin{aligned} B_n^{(r)}(f,x) &= \int_0^\infty W_n^{(r)}(t,x)f(t)dt \\ &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt \\ &\qquad + \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt \\ &=: R_1 + R_2. \end{aligned}$$

First to estimate R_1, using the binomial expansion of $(t-x)^m$, Lemma 2.2 and Remark 2.3, we have

$$R_1 = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{\nu=0}^{i}\binom{i}{\nu}(-x)^{i-\nu}\frac{\partial^r}{\partial x^r}\int_0^\infty W_n(t,x)t^\nu dt$$

$$
\begin{aligned}
&= \frac{f^{(r)}(x)}{r!}\frac{\partial^r}{\partial x^r}\int_0^\infty W_n(t,x)t^r dt \\
&= \frac{f^{(r)}(x)}{r!}\left\{\frac{(n+r)!(n-r-1)!}{n!(n-1)!}r! + \text{terms containing lower powers of } x\right\} \\
&= f^{(r)}(x) + o(1), \quad n\to\infty.
\end{aligned}
$$

Using Lemma 2.5, we obtain

$$
\begin{aligned}
R_2 &= \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt \\
&= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{Q_{i,j,r}(x)}{\{x(1+x)\}^r}\sum_{\nu=1}^\infty [\nu-(n+1)x]^j \frac{p_{n,\nu}(x)}{n} \\
&\qquad \times \int_0^\infty p_{n,\nu-1}(t)\varepsilon(t,x)(t-x)^r dt + (-1)^r\frac{(n+r)!}{(n+1)!}(1+x)^{-n-r-1}\varepsilon(0,x)(-x)^r \\
&=: R_3 + R_4.
\end{aligned}
$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$ for a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. Thus for some $M_1>0$, we can write

$$
\begin{aligned}
|R_3| \le M_1 &\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i-1}\sum_{\nu=1}^\infty p_{n,\nu}(x)\,|\nu-(n+1)x|^j\left\{\varepsilon\int_{|t-x|<\delta} p_{n,\nu-1}(t)\,|t-x|^r\,dt\right. \\
&\qquad\left. + \int_{|t-x|\ge\delta} p_{n,\nu-1}(t)M_2 t^\gamma dt\right\} \\
&=: R_5 + R_6,
\end{aligned}
$$

where

$$
M_1 = \sup_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r}
$$

and M_2 is independent of t.

Applying Schwarz's inequality for integration and summation respectively, we obtain

$$
\begin{aligned}
R_5 &\le \varepsilon M_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i-1}\sum_{\nu=1}^\infty p_{n,\nu}(x)\,|\nu-(n+1)x|^j\left(\int_0^\infty p_{n,\nu-1}(t)dt\right)^{\frac12} \\
&\qquad \times\left(\int_0^\infty p_{n,\nu-1}(t)(t-x)^{2r}dt\right)^{\frac12} \\
&\le \varepsilon M_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i}\sum_{\nu=1}^\infty p_{n,\nu}(x)\left(\frac{1}{n}\sum_{\nu=1}^\infty p_{n,\nu}(x)[\nu-(n+1)x]^{2j}\right)^{\frac12} \\
&\qquad \times\left(\frac{1}{n}\sum_{\nu=1}^\infty p_{n,\nu}(x)\int_0^\infty p_{n,\nu-1}(t)(t-x)^{2r}dt\right)^{\frac12}.
\end{aligned}
$$

Using Lemma 2.1 and Lemma 2.2, we get

$$
R_5 \le \varepsilon M_1 \mathcal{O}\left(n^{j/2}\right)\mathcal{O}\left(n^{-r/2}\right) = \varepsilon\mathcal{O}(1).
$$

Again using the Schwarz inequality, Lemma 2.1 and Corollary 2.4, we obtain

$$\begin{aligned}
R_6 &\leq M_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i-1} \sum_{\nu=1}^{\infty} p_{n,\nu}(x)\, |\nu-(n+1)x|^j \int_{|t-x|\geq\delta} p_{n,\nu-1}(t) t^{\gamma} dt \\
&\leq M_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i-1} \sum_{\nu=1}^{\infty} p_{n,\nu}(x)\, |\nu-(n+1)x|^j \left(\int_{|t-x|\geq\delta} p_{n,\nu-1}(t) dt\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{|t-x|\geq\delta} p_{n,\nu-1}(t) t^{2\gamma} dt\right)^{\frac{1}{2}} \\
&\leq M_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i} \left(\frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)[\nu-(n+1)x]^{2j}\right)^{\frac{1}{2}} \\
&\qquad \times \left(\frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} p_{n,\nu-1}(t) t^{2\gamma} dt\right)^{\frac{1}{2}} \\
&= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\mathcal{O}\left(n^{j/2}\right)\mathcal{O}\left(n^{-s/2}\right)
\end{aligned}$$

for any $s>0$.

Choosing $s>r$ we get $R_6=o(1)$. Thus, due to arbitrariness of $\varepsilon>0$, it follows that $R_3=o(1)$. Also $R_4\to 0$ as $n\to\infty$ and hence $R_2=o(1)$. Collecting the estimates of R_1 and R_2, we get the required result. □

The following result holds.

Theorem 3.2. *Let $f\in C_\gamma[0,\infty)$, $\gamma>0$. If $f^{(r+2)}$ exists at a point $x\in(0,\infty)$, then*

$$\begin{aligned}
&\lim_{n\to\infty} n[B_n^{(r)}(f,x)-f^{(r)}(x)] \\
&\qquad = r(r+1)f^{(r)}(x)+[2x(1+r)+r]f^{(r+1)}(x)+x(1+x)f^{(r+2)}(x).
\end{aligned}$$

Proof. Using Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and $\varepsilon(t,x)=\mathcal{O}\left((t-x)^{\beta}\right)$, $t\to\infty$ for some $\beta>0$. Applying Lemma 2.2, we have

$$\begin{aligned}
n[B_n^{(r)}(f,x)-f^{(r)}(x)] &= n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^{\infty} W_n^{(r)}(x,t)(t-x)^i dt - f^{(r)}(x)\right] \\
&\qquad + \left[n\int_0^{\infty} W_n^{(r)}(x,t)\varepsilon(t,x)(t-x)^{r+2}dt\right] \\
&=: E_1+E_2.
\end{aligned}$$

$$\begin{aligned}
E_1 &= n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty W_n^{(r)}(x,t)t^j dt - nf^{(r)}(x)\\
&= \frac{f^{(r)}(x)}{r!}n\left[B_n^{(r)}(t^r,x)-r!\right]+\frac{f^{(r+1)}(x)}{(r+1)!}n\left[(r+1)(-x)B_n^{(r)}(t^r,x)\right.\\
&\qquad \left.+B_n^{(r)}(t^{r+1},x)\right]+\frac{f^{(r+2)}(x)}{(r+2)!}n\left[\frac{(r+2)(r+1)}{2}x^2B_n^{(r)}(t^r,x)\right.\\
&\qquad \left. + (r+2)(-x)B_n^{(r)}(t^{r+1},x)+B_n^{(r)}(t^{r+2},x)\right].
\end{aligned}$$

Therefore by applying Remark 2.3, we get

$$\begin{aligned}
E_1 &= nf^{(r)}(x)\left[\frac{(n+r)!(n-r-1)!}{n!(n-1)!}-1\right]\\
&\quad + n\,\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)r!\left\{\frac{(n+r)!(n-r-1)!}{n!(n-1)!}\right\}\right.\\
&\quad \left. + \left\{\frac{(n+r+1)!(n-r-2)!}{n!(n-1)!}(r+1)!x + r(r+1)\frac{(n+r)!(n-r-2)!}{n!(n-1)!}r!\right\}\right]\\
&\quad + n\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)x^2}{2}\cdot r!\frac{(n+r)!(n-r-1)!}{n!(n-1)!}\right.\\
&\quad + (r+2)(-x)\left\{\frac{(n+r+1)!(n-r-2)!}{n!(n-1)!}(r+1)!x\right.\\
&\quad \left. + r(r+1)\frac{(n+r)!(n-r-2)!}{n!(n-1)!}r!\right\}\\
&\quad + \left\{\frac{(n+r+2)!(n-r-3)!}{n!(n-1)!}\;\frac{(r+2)!}{2}x^2\right.\\
&\quad \left.\left. +(r+1)(r+2)\frac{(n+r+1)!(n-r-3)!}{n!(n-1)!}(r+1)!x\right\}\right] + \mathcal{O}\left(n^{-2}\right).
\end{aligned}$$

In order to complete the proof of the theorem it is sufficient to show that $E_2 \to 0$ as $n \to \infty$, which can be easily proved along the lines of the proof of Theorem 3.1 and by using Lemma 2.1, Lemma 2.2 and Lemma 2.5. □

Let us assume that $0 < a < a_1 < b_1 < b < \infty$, for sufficiently small $\delta > 0$, the m-th order Steklov mean $f_{m,\delta}(t)$ corresponding to $f \in C_\gamma[0,\infty)$ is defined by

$$f_{m,\delta}(t) = \delta^{-m}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\cdots\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\left[f(t)-\Delta_\eta^m f(t)\right]\prod_{i=1}^{m}dt_i$$

where $\eta = \frac{1}{m}\sum_{i=1}^{m} t_i$, $t \in [a,b]$ and $\Delta_\eta^m f(t)$ is the $m-$th forward difference with step length η.

It is easily checked (see e. g. [1], [4]) that

(i) $f_{m,\delta}$ has continuous derivatives up to order m on $[a,b]$;
(ii) $\left\|f_{m,\delta}^{(r)}\right\|_{C[a_1,b_1]} \le M_1\delta^{-r}\omega_r(f,\delta,a_1,b_1)$, $r = 1,2,3,\dots,m$;
(iii) $\|f-f_{m,\delta}\|_{C[a_1,b_1]} \le M_2\omega_m(f,\delta,a,b)$;
(iv) $\|f_{m,\delta}\|_{C[a_1,b_1]} \le M_3\|f\|_\gamma$,

where M_i, for $i = 1,2,3$ are certain unrelated constants independent of f and δ. The $r-$th order modulus of continuity $\omega_r(f,\delta,a,b)$ for a function f continuous on the interval $[a,b]$ is defined by:

$$\omega_r(f,\delta,a,b) = \sup\left\{|\Delta_h^r f(x)| : |h| \le \delta;\ x, x+h \in [a,b]\right\}.$$

For $r=1$, $\omega_1(f,\delta)$ is written simply $\omega_f(\delta)$ or $\omega(f,\delta)$.

The following error estimation is in terms of higher order modulus of continuity:

Theorem 3.3. *Let $f \in C_\gamma[0,\infty)$, $\gamma > 0$ and $0 < a < a_1 < b_1 < b < \infty$. Then for all n sufficiently large*

$$\left\|B_n^{(r)}(f,*) - f^{(r)}(x)\right\|_{C[a_1,b_1]} \le \max\left\{M_3\omega_2(f^{(r)}, n^{-1/2}, a, b), M_4 n^{-1}\|f\|_\gamma\right\}$$

where $M_3 = M_3(r)$, $M_4 = M_4(r,f)$.

Proof. First by the linearity property, we have

$$\begin{aligned}\left\|B_n^{(r)}(f,*) - f^{(r)}\right\|_{C[a_1,b_1]} &\le \left\|B_n^{(r)}((f-f_{2,\delta}),*)\right\|_{C[a_1,b_1]} + \left\|B_n^{(r)}(f_{2,\delta},*) - f_{2,\delta}^{(r)}\right\|_{C[a_1,b_1]} \\ &\quad + \left\|f^{(r)} - f_{2,\delta}^{(r)}\right\|_{C[a_1,b_1]} \\ &=: A_1 + A_2 + A_3.\end{aligned}$$

By property (iii) of the Steklov mean, we have

$$A_3 \le C_1\omega_2(f^{(r)},\delta,a,b).$$

Next using Theorem 3.2, we have

$$A_2 \le C_2 n^{-(k+1)} \sum_{j=r}^{r+2} \left\|f_{2,\delta}^{(j)}\right\|_{C[a,b]}.$$

By applying the interpolation property due to Goldberg and Meir [2] for each $j = r, r+1, r+2$, we have

$$\left\|f_{2,\delta}^{(j)}\right\|_{C[a,b]} \le C_3\left\{\|f_{2,\delta}\|_{C[a,b]} + \left\|f_{2,\delta}^{(r+2)}\right\|_{C[a,b]}\right\}.$$

Therefore by applying properties (ii) and (iv) of the Steklov mean, we obtain

$$A_2 \le C_4 n^{-1}\left\{\|f\|_\gamma + \delta^{-2}\omega_2(f^{(r)},\delta)\right\}.$$

Finally we estimate A_1, choosing a^*, b^* satisfying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Also let $\psi(t)$ denote the characteristic function of the interval $[a^*, b^*]$, then

$$\begin{aligned}A_1 &\le \left\|B_n^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),*)\right\|_{C[a_1,b_1]} \\ &\quad + \left\|B_n^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t)),*)\right\|_{C[a_1,b_1]} \\ &=: A_4 + A_5.\end{aligned}$$

We may note here that to estimate A_4 and A_5, it is enough to consider their expressions without the linear combinations. By Lemma 2.6, we have

$$\begin{aligned}&B_n^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),x) \\ &\quad = \frac{(n-r-1)!(n+r-1)!}{n!(n-1)!}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^\infty p_{n-1,\nu+r-1}(t) f^{(r)}(t)\,dt.\end{aligned}$$

Hence

$$\left\|B_n^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),*)\right\|_{C[a,b]} \le C_5\left\|f^{(r)} - f_{2,\delta}^{(r)}\right\|_{C[a^*,b^*]}.$$

Now for $x \in [a_1, b_1]$ and $t \in [0, \infty) \backslash [a^*, b^*]$, we choose a $\delta_1 > 0$ satisfying $|t - x| \geq \delta_1$. Therefore by Lemma 2.5 and the Schwarz inequality, we have

$$\begin{aligned}
I &= \left|B_n^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t)),x)\right| \\
&\leq \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \\
&\qquad \times \frac{1}{n}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu-(n+1)x|^j \int_0^{\infty} p_{n,\nu-1}(t)(1-\psi(t))\,|f(t)-f_{2,\delta}(t)|\,dt \\
&\qquad + (1+x)^{-n-1}|(-n-1)(-n)\cdots(-n-r)|(1-\psi(0))\,|f(0)-f_{2,\delta}(0)| \\
&\leq C_6\,\|f\|_\gamma \left\{ \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i-1} \sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu-(n+1)x|^j \right. \\
&\qquad \left. \times \int_{|t-x|\geq\delta_1} p_{n,\nu-1}(t)dt + (1+x)^{-n-1}|(-n-1)(-n)\cdots(-n-r)| \right\} \\
&\leq C_6\,\|f\|_\gamma \left\{ \delta_1^{-2s} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i-1} \sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu-(n+1)x|^j \left(\int_0^{\infty} p_{n,\nu-1}(t)dt\right)^{\frac{1}{2}} \right. \\
&\qquad \left. \times \left(\int_0^{\infty} p_{n,\nu-1}(t)(t-x)^{4s}dt\right)^{\frac{1}{2}} + (1+x)^{-n-1}|(-n-1)(-n)\cdots(-n-r)| \right\} \\
&\leq C_6\,\|f\|_\gamma\,\delta_1^{-2s} \\
&\qquad \times \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \left\{ \frac{1}{n}\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\,[\nu-(n+1)x]^{2j} - (1+x)^{-n-1} - \{-(n+1)x\}^{2j} \right\}^{\frac{1}{2}} \\
&\qquad \times \left\{ \frac{1}{n}\sum_{\nu=0}^{\infty} p_{n,\nu}(x) \int_0^{\infty} p_{n,\nu-1}(t)(t-x)^{4s}dt - (1+x)^{-n-1}(-x)^{4s} \right. \\
&\qquad \left. - (1+x)^{-n-1}(-x)^{4s} \right\}^{\frac{1}{2}} + C_6\,\|f\|_\gamma\,(1+x)^{-n-1}|(-n-1)(-n)\cdots(-n-r)|.
\end{aligned}$$

Hence by Lemma 2.1 and Lemma 2.2, we have

$$I \leq C_7\,\|f\|_\gamma\,\delta_1^{-2s}\mathcal{O}\left(n^{(i+\frac{j}{2}-s)}\right) \leq C_7 n^{-q}\,\|f\|_\gamma\,,\ q = s - \frac{r}{2},$$

where the last term vanishes as $n \to \infty$. Now choosing q satisfying $q \geq 1$, we obtain

$$I \leq C_7 n^{-1}\,\|f\|_\gamma\,.$$

Therefore by property (iii) of the Steklov mean, we get

$$\begin{aligned}
A_1 &\leq C_8 \left\| f^{(r)} - f_{2,\delta}^{(r)} \right\|_{C[a^*,b^*]} + C_7 n^{-1}\,\|f\|_\gamma \\
&\leq C_9\omega_2(f^{(r)},\delta,a,b) + C_7 n^{-1}\,\|f\|_\gamma\,.
\end{aligned}$$

Choosing $\delta = n^{-1/2}$, the theorem follows. □

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