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A GENERALIZATION OF OZAKI-NUNOKAWA'S UNIVALENCE CRITERION

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ABSTRACT. In this paper we obtain a generalization of Ozaki-Nunokawa's univalence criterion using the method of Loewner chains.

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1. INTRODUCTION

Let A be the class of analytic functions f defined in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + a_2 z^2 + \cdots, \quad z \in U. \tag{1.1}$$

In [1] Ozaki and Nunokawa showed that if $f \in A$ and

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq |z|^2, \quad \text{for all } z \in U, \tag{1.2}$$

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then the function f is univalent in U. In this paper we use the method of Loewner chains to establish a generalization of Ozaki-Nunokawa's univalence criterion.

2. LOEWNER CHAINS AND UNIVALENCE CRITERIA

In order to prove our main result we need a brief summary of Ch. Pommerenke's method of constructing univalence criteria. A family of univalent functions

$$L(\cdot,t) : U \longrightarrow \mathbb{C}, \quad t \geq 0$$

is a Loewner chain, if $L(\cdot, s)$ is subordinate to $L(\cdot, t)$ for all $0 \leq s \leq t$. Recall that a function $f : U \longrightarrow \mathbb{C}$ is said to be subordinate to a function $g : U \longrightarrow \mathbb{C}$ (in symbols $f \prec g$) if there exists a function $\omega : U \longrightarrow U$ such that $f(z) = g(\omega(z))$ for all $z \in U$. We also recall the following known result (see [4, pp. 159–173]):

Theorem 2.1. *Let $L(z,t) = a_1(t)z + \dots$ be an analytic function of $z \in U_r = \{z \in \mathbb{C} : |z| < r\}$ for all $t \geq 0$. Suppose that:*

i) *$L(z,t)$ is a locally absolutely continuous function of t, locally uniform with respect to $z \in U_r$;*

ii) *$a_1(t)$ is a complex-valued continuous function on $[0,\infty)$ such that*

$$a_1(t) \neq 0, \qquad \lim_{t\to\infty} |a_1(t)| = \infty$$

and

$$\left\{ \frac{L(\cdot,t)}{a_1(t)} \right\}_{t\geq 0}$$

is a normal family of functions in U_r;

iii) *there exists an analytic function $p : U \times [0,\infty) \to \mathbb{C}$ satisfying*

$$\operatorname{Re} p\,(z,t) > 0, \quad \textit{for all } (z,t) \in U \times [0,\infty)$$

and

$$z\frac{\partial L(z,t)}{\partial z} = p\,(z,t)\,\frac{\partial L(z,t)}{\partial t}, \quad \textit{for any } z \in U_r, \textit{ a.e. } t \geq 0.$$

Then for all $t \geq 0$, the function $L(\cdot,t)$ has an analytic and univalent extension to the whole unit disk U.

We can now prove the main result, as follows:

Theorem 2.2. *Let $f \in A$ and let m be a positive real number such that the inequalities*

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{2.1}$$

and

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2}\,|z|^{m+1} \right| \leq \frac{m+1}{2}\,|z|^{m+1} \tag{2.2}$$

are satisfied for all $z \in U$. Then the function f is univalent in U.

Proof. Let a and b be any positive real numbers chosen such that $m = \frac{b}{a}$. We define:

$$L(z,t) = f(e^{-at}z) + \frac{\left(e^{bt} - e^{-at}\right) z \frac{f(e^{-at}z)}{(e^{-at}z)^2}}{1 - \left(e^{bt} - e^{-at}\right) z \frac{f(e^{-at}z) - e^{-at}z}{(e^{-at}z)^2}},$$

for $t \geq 0$. Since the function $f(e^{-at}z)$ is analytic in U, it is easy to see that for each $t \geq 0$ there exists an $r \in (0,1]$ arbitrarily fixed, the function $L(z,t)$ is analytic in a neighborhood U_r

of $z = 0$. If $L(z,t) = a_1(t)z + \cdots$ is the power series expansion of $L(z,t)$ in the neighborhood U_r, it can be checked that we have $a_1(t) = e^{bt}$ and therefore $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. Since $\frac{L(z,t)}{a_1(t)}$ is the summation between z and a holomorphic function, it follows that $\left\{\frac{L(\cdot,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family of functions in U_r. By elementary computations it can be shown easily that $\frac{\partial L(z,t)}{\partial z}$ can be expressed as the summation between $be^{bt}z$ and a holomorphic function. From this representation of $\frac{\partial L(z,t)}{\partial z}$ we obtain the absolute continuity requirement i) of Theorem 2.1. Let $p(z,t)$ be the function defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \bigg/ \frac{\partial L(z,t)}{\partial t}.$$

In order to prove that the function $p(z,t)$ is analytic and has a positive real part in U, we will show that the function

$$m(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \tag{2.3}$$

is analytic in U and

$$|m(z,t)| < 1 \tag{2.4}$$

for all $z \in U$ and $t \geq 0$. We have

$$m(z,t) = \frac{(1+a)F(z,t)+1-b}{(1-a)F(z,t)+1+b},$$

where

$$F(z,t) = e^{(a+b)t}\left[(e^{-at}z)^2\frac{f'(e^{-at}z)}{f^2(e^{-at}z)} - 1\right].$$

The condition (2.4) is therefore equivalent to

$$\left|F(z,t) - \frac{b-a}{2a}\right| < \frac{a+b}{2a}, \qquad \text{for all } z \in U \text{ and } t \geq 0. \tag{2.5}$$

For $t = 0$, the inequality (2.5) becomes

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2},$$

where $m = \dfrac{b}{a}$. Defining:

$$G(z,t) = e^{(a+b)t}\left[(e^{-at}z)^2\frac{f'(e^{-at}z)}{f^2(e^{-at}z)} - 1\right] - \frac{m-1}{2}$$

and observing that $|e^{-at}z| \leq e^{-at} < 1$ for all $z \in \bar{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $t > 0$, we obtain that $G(z,t)$ is an analytic function in $\bar{U}$. Using the Maximum Modulus Principle it follows that for each $t > 0$ arbitrarily fixed there exists $\theta \in \mathbb{R}$ such that:

$$|G(z,t)| < \max_{|z|=1} |G(z,t)| = \left|G(e^{i\theta},t)\right|,$$

for all $z \in U$. Let $u = e^{-at}e^{i\theta}$. We have $|u| = e^{-at}$, $e^{-(a+b)t} = (e^{-at})^{m+1} = |u|^{m+1}$, and therefore

$$\left|G(e^{i\theta},t)\right| = \left|\frac{1}{|u|^{m+1}}\left(\frac{u^2 f'(u)}{f^2(u)} - 1\right) - \frac{m-1}{2}\right|.$$

From the hypothesis (2.2) we obtain therefore:

$$\left|G(e^{i\theta},t)\right| \leq \frac{m+1}{2}. \tag{2.6}$$

From (2.1) and (2.6) it follows that the inequality (2.5) holds true for all $z \in U$ and all $t \geq 0$. Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $L(\cdot,t)$ has an analytic and univalent extension to the whole unit disk U, for all $t \geq 0$. For $t = 0$ we have $L(z,0) = f(z)$, for all $z \in U$, and therefore the function f is univalent in U, concluding the proof of the theorem. □

It is easy to check that inequality (2.2) implies the inequality (2.1) and thus we obtain the following corollary :

Corollary 2.3. *Let $f \in A$ and let m be a positive real number such that*

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \frac{m-1}{2}\,|z|^{m+1}\right| \leq \frac{m+1}{2}\,|z|^{m+1} \tag{2.7}$$

for all $z \in U$. Then the function f is univalent in U.

Remark 2.4. We conclude with the following remarks:

i) In the particular case $m = 1$, condition (2.7) of the above corollary becomes condition (1.2). Therefore, we obtain Ozaki-Nunokawa's univalence criterion as a particular case ($m = 1$) of the above corollary, which generalizes it to all positive real numbers $m > 0$.

ii) The function $f(z) = \dfrac{z}{1+z}$ satisfies the condition (2.7) of the above corollary for every positive real number $m > 0$.

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