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A GENERALIZATION OF OZAKI-NUNOKAWA'S UNIVALENCE CRITERION

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AND NICOLAE R. PASCU

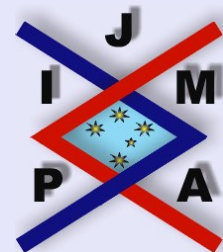
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Abstract

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Abstract

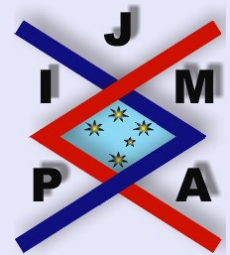
In this paper we obtain a generalization of Ozaki-Nunokawa's univalence criterion using the method of Loewner chains.

2000 Mathematics Subject Classification: 30C55.

Key words: Univalent function, univalence criteria, Loewner chains.

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1. Introduction

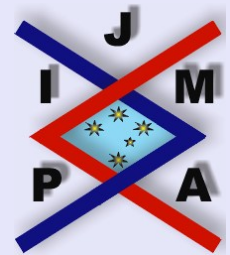
Let A be the class of analytic functions f defined in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$(1.1) \quad f(z) = z + a_2 z^2 + \dots, \quad z \in U.$$

In [1] Ozaki and Nunokawa showed that if $f \in A$ and

$$(1.2) \quad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq |z|^2, \quad \text{for all } z \in U,$$

then the function f is univalent in U . In this paper we use the method of Loewner chains to establish a generalization of Ozaki-Nunokawa's univalence criterion.



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2. Loewner Chains and Univalence Criteria

In order to prove our main result we need a brief summary of Ch. Pommerenke's method of constructing univalence criteria. A family of univalent functions

$$L(\cdot, t) : U \longrightarrow \mathbb{C}, \quad t \geq 0$$

is a Loewner chain, if $L(\cdot, s)$ is subordinate to $L(\cdot, t)$ for all $0 \leq s \leq t$. Recall that a function $f : U \longrightarrow \mathbb{C}$ is said to be subordinate to a function $g : U \longrightarrow \mathbb{C}$ (in symbols $f \prec g$) if there exists a function $\omega : U \longrightarrow U$ such that $f(z) = g(\omega(z))$ for all $z \in U$. We also recall the following known result (see [4, pp. 159–173]):

Theorem 2.1. *Let $L(z, t) = a_1(t)z + \dots$ be an analytic function of $z \in U_r = \{z \in \mathbb{C} : |z| < r\}$ for all $t \geq 0$. Suppose that:*

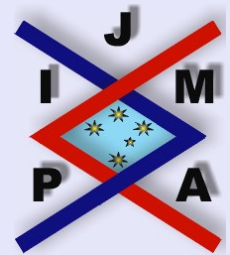
- i) $L(z, t)$ is a locally absolutely continuous function of t , locally uniform with respect to $z \in U_r$;
- ii) $a_1(t)$ is a complex-valued continuous function on $[0, \infty)$ such that

$$a_1(t) \neq 0, \quad \lim_{t \rightarrow \infty} |a_1(t)| = \infty$$

and

$$\left\{ \frac{L(\cdot, t)}{a_1(t)} \right\}_{t \geq 0}$$

is a normal family of functions in U_r ;



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iii) there exists an analytic function $p : U \times [0, \infty) \rightarrow \mathbb{C}$ satisfying

$$\operatorname{Re} p(z, t) > 0, \quad \text{for all } (z, t) \in U \times [0, \infty)$$

and

$$z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t}, \quad \text{for any } z \in U_r, \text{ a.e. } t \geq 0.$$

Then for all $t \geq 0$, the function $L(\cdot, t)$ has an analytic and univalent extension to the whole unit disk U .

We can now prove the main result, as follows:

Theorem 2.2. Let $f \in A$ and let m be a positive real number such that the inequalities

$$(2.1) \quad \left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2}$$

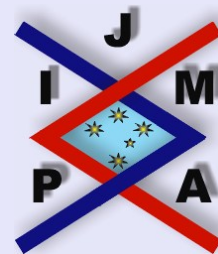
and

$$(2.2) \quad \left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2} |z|^{m+1} \right| \leq \frac{m+1}{2} |z|^{m+1}$$

are satisfied for all $z \in U$. Then the function f is univalent in U .

Proof. Let a and b be any positive real numbers chosen such that $m = \frac{b}{a}$. We define:

$$L(z, t) = f(e^{-at}z) + \frac{(e^{bt} - e^{-at}) z \frac{f(e^{-at}z)}{(e^{-at}z)^2}}{1 - (e^{bt} - e^{-at}) z \frac{f(e^{-at}z) - e^{-at}z}{(e^{-at}z)^2}},$$



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for $t \geq 0$. Since the function $f(e^{-at}z)$ is analytic in U , it is easy to see that for each $t \geq 0$ there exists an $r \in (0, 1]$ arbitrarily fixed, the function $L(z, t)$ is analytic in a neighborhood U_r of $z = 0$. If $L(z, t) = a_1(t)z + \dots$ is the power series expansion of $L(z, t)$ in the neighborhood U_r , it can be checked that we have $a_1(t) = e^{bt}$ and therefore $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$. Since $\frac{L(z,t)}{a_1(t)}$ is the summation between z and a holomorphic function, it follows that $\left\{ \frac{L(\cdot, t)}{a_1(t)} \right\}_{t \geq 0}$ is a normal family of functions in U_r . By elementary computations it can be shown easily that $\frac{\partial L(z, t)}{\partial z}$ can be expressed as the summation between $be^{bt}z$ and a holomorphic function. From this representation of $\frac{\partial L(z, t)}{\partial z}$ we obtain the absolute continuity requirement i) of Theorem 2.1. Let $p(z, t)$ be the function defined by

$$p(z, t) = z \frac{\partial L(z, t)}{\partial z} \bigg/ \frac{\partial L(z, t)}{\partial t}.$$

In order to prove that the function $p(z, t)$ is analytic and has a positive real part in U , we will show that the function

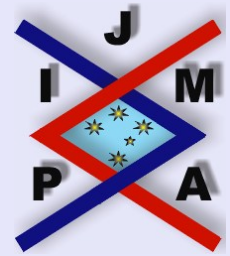
$$(2.3) \quad m(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}$$

is analytic in U and

$$(2.4) \quad |m(z, t)| < 1$$

for all $z \in U$ and $t \geq 0$. We have

$$m(z, t) = \frac{(1+a)F(z, t) + 1 - b}{(1-a)F(z, t) + 1 + b},$$



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where

$$F(z, t) = e^{(a+b)t} \left[(e^{-at}z)^2 \frac{f'(e^{-at}z)}{f^2(e^{-at}z)} - 1 \right].$$

The condition (2.4) is therefore equivalent to

$$(2.5) \quad \left| F(z, t) - \frac{b-a}{2a} \right| < \frac{a+b}{2a}, \quad \text{for all } z \in U \text{ and } t \geq 0.$$

For $t = 0$, the inequality (2.5) becomes

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2},$$

where $m = \frac{b}{a}$. Defining:

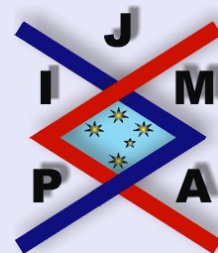
$$G(z, t) = e^{(a+b)t} \left[(e^{-at}z)^2 \frac{f'(e^{-at}z)}{f^2(e^{-at}z)} - 1 \right] - \frac{m-1}{2}$$

and observing that $|e^{-at}z| \leq e^{-at} < 1$ for all $z \in \bar{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $t > 0$, we obtain that $G(z, t)$ is an analytic function in \bar{U} . Using the Maximum Modulus Principle it follows that for each $t > 0$ arbitrarily fixed there exists $\theta \in \mathbb{R}$ such that:

$$|G(z, t)| < \max_{|z|=1} |G(z, t)| = |G(e^{i\theta}, t)|,$$

for all $z \in U$. Let $u = e^{-at}e^{i\theta}$. We have $|u| = e^{-at}$, $e^{-(a+b)t} = (e^{-at})^{m+1} = |u|^{m+1}$, and therefore

$$|G(e^{i\theta}, t)| = \left| \frac{1}{|u|^{m+1}} \left(\frac{u^2 f'(u)}{f^2(u)} - 1 \right) - \frac{m-1}{2} \right|.$$



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From the hypothesis (2.2) we obtain therefore:

$$(2.6) \quad |G(e^{i\theta}, t)| \leq \frac{m+1}{2}.$$

From (2.1) and (2.6) it follows that the inequality (2.5) holds true for all $z \in U$ and all $t \geq 0$. Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $L(\cdot, t)$ has an analytic and univalent extension to the whole unit disk U , for all $t \geq 0$. For $t = 0$ we have $L(z, 0) = f(z)$, for all $z \in U$, and therefore the function f is univalent in U , concluding the proof of the theorem. \square

It is easy to check that inequality (2.2) implies the inequality (2.1) and thus we obtain the following corollary :

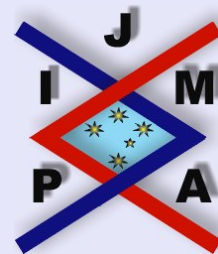
Corollary 2.3. *Let $f \in A$ and let m be a positive real number such that*

$$(2.7) \quad \left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2} |z|^{m+1} \right| \leq \frac{m+1}{2} |z|^{m+1}$$

for all $z \in U$. Then the function f is univalent in U .

Remark 2.1. *We conclude with the following remarks:*

- i) *In the particular case $m = 1$, condition (2.7) of the above corollary becomes condition (1.2). Therefore, we obtain Ozaki-Nunokawa's univalence criterion as a particular case ($m = 1$) of the above corollary, which generalizes it to all positive real numbers $m > 0$.*
- ii) *The function $f(z) = \frac{z}{1+z}$ satisfies the condition (2.7) of the above corollary for every positive real number $m > 0$.*



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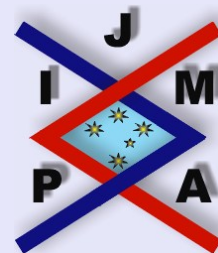
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