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INEQUALITIES FOR $J-$CONTRACTIONS INVOLVING THE $\alpha-$POWER MEAN

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ABSTRACT. A selfadjoint involutive matrix J endows $\mathbb{C}^n$ with an indefinite inner product $[\cdot,\cdot]$ given by $[x,y] := \langle Jx, y\rangle$, $x, y \in \mathbb{C}^n$. We present some inequalities of indefinite type involving the $\alpha-$power mean and the chaotic order. These results are in the vein of those obtained by E. Kamei [6, 7].

Key words and phrases: $J-$selfadjoint matrix, Furuta inequality, $J-$chaotic order, $\alpha-$power mean.

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1. INTRODUCTION

For a selfadjoint involution matrix J, that is, $J = J^*$ and $J^2 = I$, we consider $\mathbb{C}^n$ with the indefinite Krein space structure endowed by the indefinite inner product $[x,y] := y^*Jx$, $x, y \in \mathbb{C}^n$. Let M_n denote the algebra of $n\times n$ complex matrices. The $J-$*adjoint* matrix $A^\#$ of $A \in M_n$ is defined by

$$[A\,x, y] = [x, A^\#\,y], \quad x, y \in \mathbb{C}^n,$$

or equivalently, $A^\# = JA^*J$. A matrix $A \in M_n$ is said to be $J-$*selfadjoint* if $A^\# = A$, that is, if JA is selfadjoint. For a pair of $J-$selfadjoint matrices A, B, the $J-$*order relation* $A \geq^J B$ means that $[Ax, x] \geq [Bx, x]$, $x \in \mathbb{C}^n$, where this order relation means that the selfadjoint matrix $JA - JB$ is positive semidefinite. If A, B have positive eigenvalues, $\operatorname{Log}(A) \geq^J \operatorname{Log}(B)$ is called the $J-$*chaotic order*, where $\operatorname{Log}(t)$ denotes the principal branch of the logarithm function. The $J-$ chaotic order is weaker than the usual $J-$order relation $A \geq^J B$ [11, Corollary 2].

A matrix $A \in M_n$ is called a $J-$*contraction* if $I \geq^J A^\#A$. If A is $J-$selfadjoint and $I \geq^J A$, then all the eigenvalues of A are real. Furthermore, if A is a $J-$contraction, by a theorem of Potapov-Ginzburg [2, Chapter 2, Section 4], all the eigenvalues of the product $A^\#A$ are nonnegative.

Sano [11, Corollary 2] obtained the indefinite version of the Löwner-Heinz inequality of indefinite type, namely for A, B $J-$selfadjoint matrices with nonnegative eigenvalues such that $I \geq^J A \geq^J B$, then $I \geq^J A^\alpha \geq^J B^\alpha$, for any $0 \leq \alpha \leq 1$. The Löwner-Heinz inequality has

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a famous extension which is the Furuta inequality. An indefinite version of this inequality was established by Sano [10, Theorem 3.4] and Bebiano *et al.* [3, Theorem 2.1] in the following form: Let A, B be $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ (or $A \geq^J B \geq^J \mu I$) for some $\mu > 0$. For each $r \geq 0$,

$$\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{1.1}$$

and

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq^J \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{1.2}$$

hold for all $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

2. Inequalities for $\alpha-$Power Mean

For $J-$selfadjoint matrices A, B with positive eigenvalues, $A \geq^J B$ and $0 \leq \alpha \leq 1$, the $\alpha-$*power mean* of A and B is defined by

$$A \natural_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}}.$$

Since $I \geq^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ (or $I \leq^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$) the $J-$selfadjoint power $\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha}$ is well defined.

The essential part of the Furuta inequality of indefinite type can be reformulated in terms of $\alpha-$power means as follows. If A, B are $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$, then for all $p \geq 1$ and $r \geq 0$

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq^J A \tag{2.1}$$

and

$$B^{-r} \natural_{\frac{1+r}{p+r}} A^p \geq^J B. \tag{2.2}$$

The indefinite version of Kamei's satellite theorem for the Furuta inequality [7] was established in [4] as follows: If A, B are $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$, then

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq^J B \leq^J A \leq^J B^{-r} \natural_{\frac{1+r}{p+r}} A^p \tag{2.3}$$

for all $p \geq 1$ and $r \geq 0$.

Remark 1. Note that by (2.3) and using the fact that $X^{\#} A X \geq^J X^{\#} B X$ for all $X \in M_n$ if and only if $A \geq^J B$, we have $A^{1+r} \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}$ and $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \geq^J B^{1+r}$. Applying the Löwner-Heinz inequality of indefinite type, with $\alpha = \frac{1}{1+r}$, we obtain

$$A \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{p+r}} \quad \text{and} \quad \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{p+r}} \geq^J B$$

for all $p \geq 1$ and $r \geq 0$.

In [4], the following extension of Kamei's satellite theorem of the Furuta inequality was shown.

Lemma 2.1. *Let A, B be $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$A^{-r} \natural_{\frac{t+r}{p+r}} B^p \leq^J B^t \quad \textit{and} \quad A^t \leq^J B^{-r} \natural_{\frac{t+r}{p+r}} A^p,$$

for $r \geq 0$ and $0 \leq t \leq p$.

Theorem 2.2. *Let A, B be $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p \leq^J \left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}} \leq^J B \leq^J A \leq^J \left(B^{-r}\sharp_{\frac{t+r}{p+r}}A^p\right)^{\frac{1}{t}} \leq^J B^{-r}\sharp_{\frac{1+r}{p+r}}A^p, \tag{2.4}$$

for $r \geq 0$ and $1 \leq t \leq p$.

Proof. Without loss of generality, we may consider $\mu = 1$, otherwise we can replace A and B by $\frac{1}{\mu}A$ and $\frac{1}{\mu}B$. Let $1 \leq t \leq p$. Applying the Löwner Heinz inequality of indefinite type in Lemma 2.1 with $\alpha = \frac{1}{t}$, we get

$$\left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}} \leq^J B \leq^J A \leq^J \left(B^{-r}\sharp_{\frac{t+r}{p+r}}A^p\right)^{\frac{1}{t}}.$$

Let $A_1 = A$ and $B_1 = \left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}}$. Note that

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p = A^{-r}\sharp_{\frac{1+r}{t+r}}\left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right) = A_1^{-r}\sharp_{\frac{1+r}{t+r}}B_1^t. \tag{2.5}$$

Since $\mu I \geq^J A_1 \geq^J B_1$, applying Lemma 2.1 to A_1 and B_1, with $t = 1$ and $p = t$, we obtain

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p \leq^J B_1 = \left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}}.$$

The remaining inequality in (2.4) can be obtained in an analogous way using the second inequality in Lemma 2.1, with $t = 1$ and $p = t$. □

Theorem 2.3. *Let A, B be $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$\left(A^{-r}\sharp_{\frac{t_1+r}{p+r}}B^p\right)^{\frac{1}{t_1}} \leq^J \left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{1}{t_2}} \quad \textit{and} \quad \left(B^{-r}\sharp_{\frac{t_1+r}{p+r}}A^p\right)^{\frac{1}{t_1}} \geq^J \left(B^{-r}\sharp_{\frac{t_2+r}{p+r}}A^p\right)^{\frac{1}{t_2}}$$

for $r \geq 0$ and $1 \leq t_2 \leq t_1 \leq p$.

Proof. Without loss of generality, we may consider $\mu = 1$, otherwise we can replace A and B by $\frac{1}{\mu}A$ and $\frac{1}{\mu}B$. Let $A_1 = A$ and $B_1 = \left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{1}{t_2}}$. By Lemma 2.1 and the Löwner Heinz inequality of indefinite type with $\alpha = \frac{1}{t_2}$, we have $B_1 \leq^J B \leq^J A_1 \leq^J I$. Applying Lemma 2.1 to A_1 and B_1, with $p = t_2$, we obtain

$$A_1^{-r}\sharp_{\frac{t_1+r}{t_2+r}}B_1^{t_2} \leq^J B_1^{t_1} = \left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{t_1}{t_2}}. \tag{2.6}$$

On the other hand,

$$A_1^{-r}\sharp_{\frac{t_1+r}{p+r}}B^p = A_1^{-r}\sharp_{\frac{t_1+r}{t_2+r}}\left[\left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{1}{t_2}}\right]^{t_2} = A_1^{-r}\sharp_{\frac{t_1+r}{t_2+r}}B_1^{t_2}. \tag{2.7}$$

By (2.6) and (2.7),

$$A^{-r}\sharp_{\frac{t_1+r}{p+r}}B^p \leq^J \left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{t_1}{t_2}}.$$

Using the Löwner-Heinz inequality of indefinite type with $\alpha = \frac{1}{t_1}$, we have

$$\left(A^{-r}\sharp_{\frac{t_1+r}{p+r}}B^p\right)^{\frac{1}{t_1}} \leq^J \left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{1}{t_2}}.$$

The remaining inequality can be obtained analogously. □

Theorem 2.4. *Let A, B be $J-$selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$A^{-r}\natural_{\frac{t+r}{p+r}} B^p \leq^J \left(A^{-r}\natural_{\frac{1+r}{p+r}} B^p\right)^t \leq^J B^t \leq^J A^t \leq^J \left(B^{-r}\natural_{\frac{1+r}{p+r}} A^p\right)^t \leq^J B^{-r}\natural_{\frac{t+r}{p+r}} A^p$$

for $0 \leq t \leq 1 \leq p$ and $r \geq 0$.

Proof. By the indefinite version of Kamei's satellite theorem for the Furuta inequality and since $0 \leq t \leq 1$, we can apply the Löwner-Heinz inequality of indefinite type with $\alpha = t$, to get

$$\left(A^{-r}\natural_{\frac{1+r}{p+r}} B^p\right)^t \leq^J B^t \leq^J A^t \leq^J \left(B^{-r}\natural_{\frac{1+r}{p+r}} A^p\right)^t.$$

Note that

$$A^{-r}\natural_{\frac{t+r}{p+r}} B^p = \left(A^t\right)^{-\frac{r}{t}} \natural_{\frac{t+r}{1+r}} \left[\left(A^{-r}\natural_{\frac{1+r}{p+r}} B^p\right)^t\right]^{\frac{1}{t}}.$$

Since $\mu I \geq^J A^t$, for all $t > 0$ [10] and $A^t \geq^J \left(A^{-r}\natural_{\frac{1+r}{p+r}} B^p\right)^t$, applying the indefinite version of Kamei's satellite theorem for the Furuta inequality with A and B replaced by A^t and $\left(A^{-r}\natural_{\frac{1+r}{p+r}} B^p\right)^t$, respectively, and with r replaced by r/t and p replaced by $1/t$, we have

$$A^{-r}\natural_{\frac{t+r}{p+r}} B^p \leq^J \left(A^{-r}\natural_{\frac{1+r}{p+r}} B^p\right)^t.$$

The remaining inequality can be obtained analogously. □

3. Inequalities Involving the $J-$Chaotic Order

The following theorem is the indefinite version of the *Chaotic Furuta inequality*, a result previously stated in the context of Hilbert spaces by Fujii, Furuta and Kamei [5].

Theorem 3.1. *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then the following statements are mutually equivalent:*

(i) $\operatorname{Log}(A) \geq^J \operatorname{Log}(B)$;

(ii) $\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \leq^J A^r$, *for all $p \geq 0$ and $r \geq 0$;*

(iii) $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq^J B^r$, *for all $p \geq 0$ and $r \geq 0$.*

Under the chaotic order $\operatorname{Log}(A) \geq^J \operatorname{Log}(B)$, we can obtain the satellite theorem of the Furuta inequality. To prove this result, we need the following lemmas.

Lemma 3.2 ([10])**.** *If A, B are $J-$selfadjoint matrices with positive eigenvalues and $A \geq^J B$, then $B^{-1} \geq^J A^{-1}$.*

Lemma 3.3 ([10])**.** *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $I \geq^J A$, $I \geq^J B$. Then*

$$(ABA)^{\lambda} = AB^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^2B^{\frac{1}{2}}\right)^{\lambda-1} B^{\frac{1}{2}}A, \qquad \lambda \in \mathbb{R}.$$

Theorem 3.4 (Satellite theorem of the chaotic Furuta inequality)**.** *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. If $\operatorname{Log}(A) \geq^J \operatorname{Log}(B)$ then*

$$A^{-r}\natural_{\frac{1+r}{p+r}} B^p \leq^J B \quad \text{and} \quad B^{-r}\natural_{\frac{1+r}{p+r}} A^p \geq^J A$$

for all $p \geq 1$ and $r \geq 0$.

Proof. Let $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$. Interchanging the roles of r and p in Theorem 3.1 from the equivalence between (i) and (iii), we obtain

$$\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \geq^J B^p, \tag{3.1}$$

for all $p \geq 0$ and $r \geq 0$. From Lemma 3.3, we get

$$A^{-\frac{r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}A^{-\frac{r}{2}} = B^{\frac{p}{2}}\left[\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{-\frac{p}{p+r}}\right]^{\frac{p-1}{p}}B^{\frac{p}{2}}.$$

Hence, applying Lemma 3.2 to (3.1), noting that $0 \leq (p-1)/p \leq 1$ and using the Löwner-Heinz inequality of indefinite type, we have

$$A^{-\frac{r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}A^{-\frac{r}{2}} \leq^J B^{\frac{p}{2}}B^{1-p}B^{\frac{p}{2}} = B.$$

The result now follows easily. The remaining inequality can be analogously obtained. □

As a generalization of Theorem 3.4, we can obtain the next characterization of the chaotic order.

Theorem 3.5. *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then the following statements are equivalent:*

(i) $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$;
(ii) $A^{-r}\natural_{\frac{t+r}{p+r}}B^p \leq^J B^t$, *for* $r \geq 0$ *and* $0 \leq t \leq p$;
(iii) $B^{-r}\natural_{\frac{t+r}{p+r}}A^p \geq^J A^t$, *for* $r \geq 0$ *and* $0 \leq t \leq p$;
(iv) $A^{-r}\natural_{\frac{-t+r}{p+r}}B^p \leq^J A^{-t}$, *for* $r \geq 0$ *and* $0 \leq t \leq r$;
(v) $B^{-r}\natural_{\frac{-t+r}{p+r}}A^p \geq^J B^{-t}$, *for* $r \geq 0$ *and* $0 \leq \delta \leq r$.

Proof. We first prove the equivalence between (i) and (iv). By Theorem 3.1, $\mathrm{Log}(A) \geq^J \mathrm{Log}(B)$ is equivalent to $\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \leq^J A^r$, for all $p \geq 0$ and $r \geq 0$. Henceforth, since $0 \leq t \leq r$ applying the Löwner-Heinz inequality of indefinite type, we easily obtain

$$\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{-t+r}{p+r}} = \left[\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}}\right]^{\frac{-t+r}{r}} \leq^J A^{r-t}.$$

Analogously, using the equivalence between (i) and (iii) in Theorem 3.1, we easily obtain that (i) is equivalent to (v).

(ii)⇔ (v) Suppose that (ii) holds. By Lemma 3.3 and using the fact that $X^\#AX \geq^J X^\#BX$ for all $X \in M_n$ if and only if $A \geq^J B$, we have

$$\begin{aligned} A^{\frac{r}{2}}B^tA^{\frac{r}{2}} &\geq^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}} \\ &= A^{\frac{r}{2}}B^{\frac{p}{2}}\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{t-p}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}}. \end{aligned}$$

It easily follows by Lemma 3.2, that

$$B^{p-t} \leq^J \left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{-t+p}{p+r}},$$

for $r \geq 0$ and $0 \leq t \leq p$. Replacing p by r, we obtain (v).

In an analogous way, we can prove that (v)⇔(iii). □

Remark 2. Consider two $J-$selfadjoint matrices A, B with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Let $1 \leq t \leq p$. Applying the Löwner Heinz inequality of indefinite type in Theorem 3.5 (ii) with $\alpha = \frac{1}{t}$, we obtain that $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$ if and only if

$$\left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}} \leq^J B.$$

Consider $A_1 = A$ and $B_1 = \left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}}$. Following analogous steps to the proof of Theorem 2.2 we have

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p = A_1^{-r}\sharp_{\frac{1+r}{t+r}}B_1^t.$$

Since $B_1 \leq^J B \leq^J \mu I$ and $A_1 \leq^J \mu I$, applying Theorem 3.5 (ii) to A_1 and B_1, with $t = 1$ and $p = t$, we obtain $\mathrm{Log}\,(A_1) \geq^J \mathrm{Log}\,(B_1)$ if and only if

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p \leq^J \left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}}.$$

Note that $\mathrm{Log}\,(A_1) \geq^J \mathrm{Log}\,(B_1)$ is equivalent to $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$, when $r \longrightarrow 0^+$. In this way we can easily obtain Corollary 3.6, Corollary 3.8 and Corollary 3.8 from Theorem 3.5:

Corollary 3.6. *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$ if and only if*

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p \leq^J \left(A^{-r}\sharp_{\frac{t+r}{p+r}}B^p\right)^{\frac{1}{t}} \leq^J B \quad \textit{and} \quad A \leq^J \left(B^{-r}\sharp_{\frac{t+r}{p+r}}A^p\right)^{\frac{1}{t}} \leq^J B^{-r}\sharp_{\frac{1+r}{p+r}}A^p,$$

for $r \geq 0$ and $1 \leq t \leq p$.

Corollary 3.7. *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$ if and only if*

$$\left(A^{-r}\sharp_{\frac{t_1+r}{p+r}}B^p\right)^{\frac{1}{t_1}} \leq^J \left(A^{-r}\sharp_{\frac{t_2+r}{p+r}}B^p\right)^{\frac{1}{t_2}} \quad \textit{and} \quad \left(B^{-r}\sharp_{\frac{t_1+r}{p+r}}A^p\right)^{\frac{1}{t_1}} \geq^J \left(B^{-r}\sharp_{\frac{t_2+r}{p+r}}A^p\right)^{\frac{1}{t_2}}$$

for $r \geq 0$ and $1 \leq t_2 \leq t_1 \leq p$.

Corollary 3.8. *Let A, B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then $\mathrm{Log}\,(A) \geq^J \mathrm{Log}\,(B)$ if and only if*

$$A^{-r}\sharp_{\frac{t+r}{p+r}}B^p \leq^J \left(A^{-r}\sharp_{\frac{1+r}{p+r}}B^p\right)^{t} \leq^J B^t \quad \textit{and} \quad A^t \leq^J \left(B^{-r}\sharp_{\frac{1+r}{p+r}}A^p\right)^{t} \leq^J B^{-r}\sharp_{\frac{t+r}{p+r}}A^p,$$

for $r \geq 0$ and $0 \leq t \leq 1 \leq p$.

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