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A REFINEMENT OF AN INEQUALITY FROM INFORMATION THEORY

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ABSTRACT. We discuss a refinement of an inequality from Information Theory using other well known inequalities. Then we consider relationships between the logarithmic mean and inequalities of the geometric-arithmetic means.

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1. RESULTS

The following inequality is well known in Information Theory [1], see also [4].

Proposition 1.1. *Let* $p_i\,, g_i > 0$, *where* $1 \le i \le n$ *and* $\sum_{i=1}^n p_i = \sum_{i=1}^n g_i$. *Then* $0 \le \sum_{i=1}^n p_i \ln(p_i/g_i)$ *with equality iff* $p_i = g_i$, *for all* i.

The following improves this inequality. Indeed, the lower bound is sharpened, an upper bound is provided, and the equality condition is built right in.

Proposition 1.2. *Let* $p_i\,, g_i > 0$, *where* $1 \le i \le n$ *and* $\sum_{i=1}^n p_i = \sum_{i=1}^n g_i$. *Then the following estimates hold.*

$$\sum_{i=1}^n \frac{g_i(g_i-p_i)^2}{(g_i)^2+(\max(g_i,p_i))^2} \le \sum_{i=1}^n p_i \ln\left(\frac{p_i}{g_i}\right) \le \sum_{i=1}^n \frac{g_i(g_i-p_i)^2}{(g_i)^2+(\min(g_i,p_i))^2}\,.$$

Proof. We begin with the inequality [6]

$$\frac{1}{x^2+1} \le \frac{\ln(x)}{x^2-1} \le \frac{1}{2x}, \text{ for } x>0. \tag{1.1}$$

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Thus

$$\frac{x^2-1}{2x} \le \ln(x) \le \frac{x^2-1}{x^2+1} \qquad \text{for } 0 < x \le 1\,,$$

and

$$\frac{x^2-1}{x^2+1} \le \ln(x) \le \frac{x^2-1}{2x} \qquad \text{for } 1 < x\,.$$

Equalities occur only for $x = 1$. We rewrite these as

$$x - 1 - \frac{(x-1)^2}{2x} \le \ln(x) \le x - 1 - \frac{x(x-1)^2}{x^2+1} \qquad \text{for } 0 < x \le 1\,, \tag{1.2}$$

and

$$x - 1 - \frac{x(x-1)^2}{x^2+1} \le \ln(x) \le x - 1 - \frac{(x-1)^2}{2x} \qquad \text{for } 1 < x\,. \tag{1.3}$$

Now, substituting g_i/p_i for x in (1.2) and (1.3), and then summing we obtain

$$\begin{aligned} \sum_{g_i \le p_i} g_i - \sum_{g_i \le p_i} p_i - \sum_{g_i \le p_i} \frac{g_i(g_i-p_i)^2}{(g_i)^2+(g_i)^2} &\le \sum_{g_i \le p_i} p_i \ln\left(\frac{g_i}{p_i}\right) \\ &\le \sum_{g_i \le p_i} g_i - \sum_{g_i \le p_i} p_i - \sum_{g_i \le p_i} \frac{g_i(g_i-p_i)^2}{(g_i)^2+(p_i)^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{g_i > p_i} g_i - \sum_{g_i > p_i} p_i - \sum_{g_i > p_i} \frac{g_i(g_i-p_i)^2}{(g_i)^2+(p_i)^2} &\le \sum_{g_i > p_i} p_i \ln\left(\frac{g_i}{p_i}\right) \\ &\le \sum_{g_i > p_i} g_i - \sum_{g_i > p_i} p_i - \sum_{g_i > p_i} \frac{g_i(g_i-p_i)^2}{(g_i)^2+(g_i)^2} \end{aligned}$$

respectively.

Taking these together and using $\sum_{i=1}^n p_i = \sum_{i=1}^n g_i$ we have our proposition. □

2. REMARKS

Remark 2.1. With $G = \sqrt{xy}$, $L = (x-y)/(\ln(x) - \ln(y))$, and $A = (x+y)/2$, being the Geometric, Logarithmic, and Arithmetic Means of $x, y > 0$ respectively, the inequality $G \le L \le A$ is well known [8], [2]. This can be proved by observing (c.f. [5]) that

$$L = \int_0^1 x^t y^{1-t}\,dt,$$

and then applying the following:

Theorem 2.2 (Hadamard's Inequality). *If f is a convex function on $[a,b]$, then*

$$(b-a) f\left(\frac{a+b}{2}\right) \le \int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}(b-a)$$

with the inequalities being strict when f is not constant.

The inequality in (1.1) now can be obtained by letting $y = 1/x$ in $G \le L \le A$. Thus any refinement of $G \le L \le A$ would lead to an improved version of (1.1) and, in principle, to an improvenemt of Proposition 1.2. For example, it is also known that $G \le G^{\frac{2}{3}} A^{\frac{1}{3}} \le L \le \frac{2}{3}G + \frac{1}{3}A \le A$ [3], [8], [2]. The latter can be proved simply by observing that the left side of Hadamard's Inequality is the midpoint approximation M to L and the right side is the trapezoid approximation T. Now $\frac{2}{3}M + \frac{1}{3}T$ is Simpson's rule and looking at the error term there (e.g. [7]) yields $L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$.

Remark 2.3. Using $G \leq G^{\frac{2}{3}}A^{\frac{1}{3}} \leq L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$, with $y = x + 1$ we get

$$\sqrt{x(x+1)} \leq (\sqrt{x(x+1)})^{\frac{2}{3}} \left(\frac{2x+1}{2}\right)^{\frac{1}{3}} \leq \frac{1}{\ln(1+\frac{1}{x})} \leq \frac{2}{3}\sqrt{x(x+1)} + \frac{1}{3}\frac{2x+1}{2} \leq \frac{2x+1}{2}.$$

Therefore

$$\left(1+\frac{1}{x}\right)^{\frac{2}{3}\sqrt{x(x+1)}+\frac{1}{3}\frac{2x+1}{2}} < e < \left(1+\frac{1}{x}\right)^{(\sqrt{x(x+1)})^{2/3}\left(\frac{2x+1}{2}\right)^{1/3}}$$

(c.f. [4]). For example $x = 100$ gives $2.71828182842204 < e < 2.71828182846830$. Now $e = 2.71828182845905\ldots$, so the left and right hand sides are both correct to 10 decimal places. We point out also that x does not need to be an integer.

Remark 2.4. Using $G \leq G^{\frac{2}{3}}A^{\frac{1}{3}} \leq L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$, and replacing x with e^x and letting $y = e^{-x}$, we have

$$1 \leq (\cosh(x))^{1/3} \leq \frac{\sinh(x)}{x} \leq \frac{2}{3} + \frac{1}{3}\cosh(x) \leq \cosh(x).$$

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