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ON THE L^1 NORM OF THE WEIGHTED MAXIMAL FUNCTION OF FEJÉR KERNELS WITH RESPECT TO THE WALSH-KACZMARZ SYSTEM

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ABSTRACT. The main aim of this paper is to investigate the integral of the weighted maximal function of the Walsh-Kaczmarz-Fejér kernels. We give a necessary and sufficient conditions for that the weighted maximal function of the Walsh-Kaczmarz-Fejér kernels is in L^1. After this we discuss the weighted maximal function of (C, α) kernels with respect to Walsh-Paley system too.

Key words and phrases: Walsh-Kaczmarz system, Fejér kernels, Fejér means, Maximal operator.

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1. INTRODUCTION AND PRELIMINARIES

The Walsh-Kaczmarz system was introduced in 1948 by Šneider [9]. He showed that the behavior of the Dirichlet kernel of the Walsh-Kaczmarz system is worse than of the kernel of the Walsh-Paley system. Namely, he showed in [9] that the inequality $\limsup \frac{|D_n(x)|}{\log n} \geq C > 0$ holds a.e. for the Dirichlet kernel with respect to the Walsh-Kaczmarz system. This allows us to construct examples of divergent Fourier series [2].

On the other hand, Schipp [6] and Wo-Sang Young [10] proved that the Walsh-Kaczmarz system is a convergence system. Skvorcov [8] verified the everywhere and uniform convergence of the Fejér means for continous functions. Gát proved [4] that the Fejér-Lebesgue theorem holds for the Walsh-Kaczmarz system.

It is easy to show that the L^1 norm of $\sup_n |D_n|$ with respect to both systems is infinite. Gát in [3] raised the following problem: "What happens if we apply some weight function α? That is, on what conditions do we find the inequality

$$\left\| \sup_n \left| \frac{D_n}{\alpha(n)} \right| \right\|_1 < \infty$$

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to be valid?" He gave necessary and sufficient conditions for both rearrangements of the Walsh system. The main aim of this paper to give necessary and sufficient conditions for the maximal function of Fejér kernels with weight function α for both rearrangements.

First we give a brief introduction to the theory of dyadic analysis [7, 1].

Denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, the group operation is modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is $1/2$, that is, $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let

$$G := \mathop{\times}_{k=0}^{\infty} \mathbf{Z}_2,$$

G is called the Walsh group. The elements of G can be represented by a sequence $x = (x_0, x_1, \dots, x_k, \dots)$, where $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$ $(\mathbf{N} := \{0, 1, \dots\}, \mathbf{P} := \mathbf{N} \setminus \{0\})$.

The group operation on G is coordinate-wise addition (denoted by $+$), the measure (denoted by μ) and the topology are the product measure and topology. Consequently, G is a compact Abelian group. Dyadic intervals are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1} \dots)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Furthermore, let $L^p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\| \cdot \|$). The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i, \ \ n_i \in \{0, 1\}$ $(i \in \mathbf{N})$, where only a finite number of n_i's are different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. That is, $|n|$ is the integer part of the binary logarithm of n.

Define the Walsh-Paley functions by

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Let the Walsh-Kaczmarz functions be defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Walsh-Paley system is $\omega := (\omega_n : n \in \mathbf{N})$ and the Walsh-Kaczmarz system is $\kappa := (\kappa_n : n \in \mathbf{N})$. It is well known that

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{N}$ and $\kappa_0 = \omega_0$.

A relation between Walsh-Kaczmarz functions and Walsh-Paley functions was given by Skvorcov in the following way [8]. Let the transformation $\tau_A : G \to G$ be defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots)$$

for $A \in \mathbf{N}$. We have that

$$\kappa_n(x) = r_{|n|}(x) \omega_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

Define the Dirichlet and Fejér kernels by

$$D_n^{\phi} := \sum_{k=0}^{n-1} \phi_k, \qquad K_n^{\phi} := \frac{1}{n} \sum_{k=1}^{n} D_k^{\phi},$$

where $\phi_n = \omega_n$ or κ_n $(n \in \mathbf{P})$. $D_0^\phi, K_0^\phi := 0$.

It is known [7] that

$$D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbf{N}). \end{cases}$$

Let $\alpha, \beta : [0, \infty) \to [1, \infty)$ be monotone increasing functions and define the weighted maximal function of the Dirichlet kernels $D_\alpha^{\phi,*}$ and of the Fejér kernels $K_\alpha^{\phi,*}$:

$$D_\alpha^{\phi,*}(x) := \sup_{n\in\mathbf{N}} \frac{|D_n^\phi(x)|}{\alpha([\log n])}, \qquad K_\alpha^{\phi,*}(x) := \sup_{n\in\mathbf{N}} \frac{|K_n^\phi(x)|}{\alpha([\log n])} \quad (x \in G),$$

where ϕ is either the Walsh-Paley, or the Walsh-Kaczmarz system. For the the weighted maximal function of the Dirichlet kernels with respect to the Walsh-Paley system $D_\alpha^{\omega,*}$ Gát [3] proved that $D_\alpha^{\omega,*} \in L^1$ if and only if $\sum_{A=0}^\infty \frac{1}{\alpha(A)} < \infty$. Moreover, he proved that

$$\frac{1}{2}\sum_{A=0}^\infty \frac{1}{\alpha(A)} \le \|D_\alpha^{\omega,*}\|_1 \le 2\sum_{A=0}^\infty \frac{1}{\alpha(A)}.$$

For the Walsh-Kaczmarz system, he showed that the situation is changed, namely $D_\alpha^{\kappa,*} \in L^1$ if and only if $\sum_{A=1}^\infty \frac{A}{\alpha(A)} < \infty$. Moreover, he proved that there exists a positive constant C such that

$$\|D_\alpha^{\kappa,*}\|_1 \ge \frac{1}{25}\sum_{A=1}^\infty \frac{A}{\alpha(A)} - C.$$

The two conditions are quite different for the two rearrangements of the Walsh system.

2. THE RESULTS

For $\|K_\alpha^{\omega,*}\|_1$, we immediately obtain from Gát's result the following lemma:

Lemma 2.1. *$K_\alpha^{\omega,*} \in L^1$ if and only if $\sum_{A=0}^\infty \frac{1}{\alpha(A)} < \infty$. Moreover,*

$$\frac{1}{4}\sum_{A=0}^\infty \frac{1}{\alpha(A)} \le \|K_\alpha^{\omega,*}\|_1 \le 2\sum_{A=0}^\infty \frac{1}{\alpha(A)}.$$

Proof. The upper estimation follows trivially from

$$\frac{|K_n^\omega(x)|}{\alpha(|n|)} \le \frac{1}{n}\sum_{j=1}^n \frac{|D_j^\omega(x)|}{\alpha(|j|)} \le \frac{1}{n}\sum_{j=1}^n D_\alpha^{\omega,*}(x) \le D_\alpha^{\omega,*}(x),$$

that is

$$K_\alpha^{\omega,*}(x) \le D_\alpha^{\omega,*}(x) \quad (x \in G).$$

The lower estimation for $\phi = \omega$ or κ comes from the following. On the set $I_A \backslash I_{A+1}$ we have

$$K_{2^A}^\phi(x) = \frac{1}{2^A}\sum_{k=1}^{2^A} k = \frac{2^A+1}{2}.$$

Thus, we have

$$\|K_\alpha^{\phi,*}\|_1 = \sum_{A=0}^\infty \int_{I_A\backslash I_{A+1}} K_\alpha^{\phi,*}(x)d\mu(x) \ge \sum_{A=0}^\infty \int_{I_A\backslash I_{A+1}} \frac{K_{2^A}^\phi(x)}{\alpha(A)}d\mu(x)$$
$$= \sum_{A=0}^\infty \frac{1}{\alpha(A)}\int_{I_A\backslash I_{A+1}} \frac{2^A+1}{2}d\mu(x) \ge \frac{1}{4}\sum_{A=0}^\infty \frac{1}{\alpha(A)}.$$

□

We will show that we can obtain as good an estimation for $\|K_\alpha^{\kappa,*}\|_1$ as for $\|K_\alpha^{\omega,*}\|_1$. This means that the behavior of the Walsh-Kaczmarz-Fejér kernels is better than the behavior of the Walsh-Kaczmarz-Dirichlet kernels. This is the main reason, why we have so many convergence theorems for Walsh-Kaczmarz-Fejér means [4, 8]. Namely,

Theorem 2.2. *There is positive absolute constant C such that*

$$\frac{1}{4}\sum_{A=0}^{\infty}\frac{1}{\alpha(A)} \le \|K_\alpha^{\kappa,*}\|_1 \le C\sum_{A=0}^{\infty}\frac{1}{\alpha(A)}.$$

Corollary 2.3. $K_\alpha^{\kappa,*}\in L^1$ *if and only if* $\sum_{A=0}^{\infty}\frac{1}{\alpha(A)}<\infty$.

Skvorcov in [8] proved that for $n\in\mathbf{P}$, $x\in G$

$$nK_n^{\kappa}(x) = 1 + \sum_{i=0}^{|n|-1}2^iD_{2^i}(x) + \sum_{i=0}^{|n|-1}2^ir_i(x)K_{2^i}^{\omega}(\tau_i(x)) + (n-2^{|n|})(D_{2^{|n|}}(x) + r_{|n|}(x)K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))).$$

To prove Theorem 2.2, we will use two lemmas by Gát [4].

Lemma 2.4. *Let $A,t\in\mathbf{N}$, $A>t$. Suppose that $x\in I_t\backslash I_{t+1}$. Then*

$$K_{2^A}^{\omega}(x) = \begin{cases} 0 & \text{if } x-x_te_t\notin I_A,\\ 2^{t-1} & \text{if } x-x_te_t\in I_A.\end{cases}$$

If $x\in I_A$, then $K_{2^A}^{\omega}(x)=\frac{2^A+1}{2}$.

Set

$$K_{a,b}^{\omega} := \sum_{j=a}^{a+b-1}D_j^{\omega} \qquad (a,b\in\mathbf{N}),$$

and $n^{(s)} := \sum_{i=s}^{\infty}n_i2^i$ $(n,s\in\mathbf{N})$. Using simple calculations, we have

$$nK_n^{\omega} = \sum_{s=0}^{|n|}n_sK_{n^{(s+1)},2^s}^{\omega} + D_n^{\omega} \qquad (n\in\mathbf{P}).$$

Lemma 2.5. *Let $s,t,n\in\mathbf{N}$, and $x\in I_t\backslash I_{t+1}$. If $s\le t\le|n|$, then $|K_{n^{(s+1)},2^s}^{\omega}(x)|\le c2^{s+t}$. If $t<s\le|n|$, then we have*

$$K_{n^{(s+1)},2^s}^{\omega}(x) = \begin{cases} 0 & \text{if } x-x_te_t\notin I_s,\\ \omega_{n^{(s+1)}}(x)2^{s+t-1} & \text{if } x-x_te_t\in I_s.\end{cases}$$

Throughout the remainder of the paper C will denote a positive absolute constant, though not always the same at different occurences.

Proof of the Theorem 2.2. We will use Skvorcov's result and

$$\frac{1}{n\alpha(|n|)} + \frac{1}{n\alpha(|n|)}\sum_{i=0}^{|n|-1}2^iD_{2^i}(x) + \frac{1}{n\alpha(|n|)}(n-2^{|n|})D_{2^{|n|}}(x) \le \frac{1}{\alpha(1)} + \frac{1}{n}\sum_{i=0}^{|n|-1}2^i\frac{D_{2^i}(x)}{\alpha(i)} + D_\alpha^{\omega,*}(x) \le \frac{1}{\alpha(1)} + CD_\alpha^{\omega,*}(x).$$

Now, we discuss

$$\frac{1}{n\alpha(|n|)}\sum_{i=0}^{|n|-1}2^i r_i(x)K_{2^i}^{\omega}(\tau_i(x)).$$

Let $J_t^i := \{x \in G : x_{i-1} = \cdots = x_{i-t} = 0, x_{i-t-1} = 1\}$ and $J_0^i := \{x \in G : x_{i-1} = 1\}$. For every $1 \leq i \in \mathbf{N}$ we can decompose G as the disjoint union: $G := I_i \cup \bigcup_{t=0}^{i-1} J_t^i$.

By Gát's Lemma 2.4, if $x \in J_t^i$, then $K_{2^i}^{\omega}(\tau_i(x)) \neq 0$ only in the case when $x_{i-t-2} = \cdots = x_0 = 0$, and in this case $K_{2^i}^{\omega}(\tau_i(x)) = 2^{t-1}$.

$$\begin{aligned}\int_G |r_i(x)K_{2^i}^{\omega}(\tau_i(x))|d\mu(x) &= \int_{I_i} K_{2^i}^{\omega}(\tau_i(x))d\mu(x) + \int_{\overline{I_i}} K_{2^i}^{\omega}(\tau_i(x))d\mu(x)\\ &\leq \frac{2^i+1}{2}\cdot\frac{1}{2^i} + \sum_{t=0}^{i-1}\int_{J_t^i} K_{2^i}^{\omega}(\tau_i(x))d\mu(x)\\ &\leq 1 + \sum_{t=0}^{i-1}\int_{\{x\in G: x_{i-t-1}=1, x_j=0 \text{ if } j<i \text{ and } j\neq i-t-1\}} 2^{t-1}d\mu(x)\\ &\leq 1 + \sum_{t=0}^{i-1}\frac{2^{t-1}}{2^i} \leq 2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left\|\sup_n\left|\frac{1}{n\alpha(|n|)}\sum_{i=0}^{|n|-1}2^i r_i(x)K_{2^i}^{\omega}(\tau_i(x))\right|\right\|_1 &\leq \sum_{q=0}^{\infty}\int_G \sup_{|n|=q}\frac{1}{2^q\alpha(q)}\sum_{i=0}^{q-1}2^i|r_i(x)K_{2^i}^{\omega}(\tau_i(x))|d\mu(x)\\ &\leq \sum_{q=0}^{\infty}\frac{1}{2^q\alpha(q)}\sum_{i=0}^{q-1}2^i\int_G |r_i(x)K_{2^i}^{\omega}(\tau_i(x))|d\mu(x)\\ &\leq \sum_{q=0}^{\infty}\frac{1}{2^q\alpha(q)}\sum_{i=0}^{q-1}2^{i+1} \leq C\sum_{q=0}^{\infty}\frac{1}{\alpha(q)}.\end{aligned}$$

We have to discuss

$$\sup_n\left|\frac{n-2^{|n|}}{n\alpha(|n|)}r_{|n|}(x)K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))\right|.$$

$$\begin{aligned}&\int_G \sup_n\left|\frac{n-2^{|n|}}{n\alpha(|n|)}r_{|n|}(x)K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))\right|d\mu(x)\\ &\leq \sum_{l=1}^{\infty}\frac{1}{\alpha(l)}\int_G \sup_{|n|=l}\frac{n-2^{|n|}}{n}\left|K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))\right|d\mu(x)\\ &= \sum_{l=1}^{\infty}\frac{1}{\alpha(l)}\int_{I_l} \sup_{|n|=l}\frac{n-2^{|n|}}{n}\left|K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))\right|d\mu(x)\\ &\qquad + \sum_{l=1}^{\infty}\frac{1}{\alpha(l)}\int_{\overline{I_l}} \sup_{|n|=l}\frac{n-2^{|n|}}{n}\left|K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))\right|d\mu(x)\\ &=: S^1 + S^2.\end{aligned}$$

If $x \in I_{|n|}$, then $\tau_{|n|}(x) \in I_{|n|}$ and $\left|K^{\omega}_{n-2^{|n|}}(\tau_{|n|}(x))\right| \le C(n-2^{|n|})$ and

$$\begin{aligned} S^1 &\le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \int_{I_l} \sup_{|n|=l} \frac{(n-2^{|n|})^2}{n} d\mu(x) \\ &\le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \int_{I_l} \sup_{|n|=l} (n-2^{|n|}) d\mu(x) \\ &\le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \int_{I_l} 2^l d\mu(x) \le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)}. \end{aligned}$$

Now, we investigate S^2.

$$\begin{aligned} S^2 &\le \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \int_{J_t^l} \sup_{\substack{|n|=l \\ q<l}} \sup_{|n-2^{|n|}|=q} \frac{n-2^{|n|}}{n} \left|K^{\omega}_{n-2^{|n|}}(\tau_{|n|}(x))\right| d\mu(x) \\ &\le \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \int_{J_t^l} \sup_{\substack{|n|=l \\ q<l}} \sup_{|n-2^{|n|}|=q} \frac{1}{n} \sum_{s=0}^{q} n_s \left|K^{\omega}_{n^{(s+1)},2^s}(\tau_{|n|}(x))\right| d\mu(x) \\ &\qquad + \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \int_{J_t^l} \sup_{\substack{|n|=l \\ q<l}} \sup_{|n-2^{|n|}|=q} \frac{1}{n} \left|D^{\omega}_{n-2^{|n|}}(\tau_{|n|}(x))\right| d\mu(x) \\ &=: \sum_K + \sum_D . \end{aligned}$$

Let $x \in J_t^l$. By Lemma 2.5 of Gát, if $s \le t$, then $\left|K^{\omega}_{n^{(s+1)},2^s}(\tau_{|n|}(x))\right| \le 2^{s+t}$, if $q \ge s > t$, then $K^{\omega}_{n^{(s+1)},2^s}(\tau_{|n|}(x)) \ne 0$ if and only if $x_{l-t-2} = \cdots = x_{l-s} = 0$, and in this case $\left|K^{\omega}_{n^{(s+1)},2^s}(\tau_{|n|}(x))\right| = 2^{s+t}$.

$$\begin{aligned} \sum_K &\le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \int_{J_t^l} \sup_{|n|=l} \sum_{q=0}^{l-1} \frac{1}{2^l+2^q} \sum_{s=0}^{q} \left|K^{\omega}_{n^{(s+1)},2^s}(\tau_{|n|}(x))\right| d\mu(x) \\ &\le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \sum_{q=0}^{t} \frac{1}{2^l+2^q} \sum_{s=0}^{q} \int_{J_t^l} 2^{s+t} d\mu(x) \\ &\qquad + C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \sum_{q=t+1}^{l-1} \frac{1}{2^l+2^q} \sum_{s=0}^{t} \int_{J_t^l} 2^{s+t} d\mu(x) \\ &\qquad + C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \sum_{q=t+1}^{l-1} \frac{1}{2^l+2^q} \sum_{s=t+1}^{q} \int_{\{x\in J_t^l : x_{l-t-2}=\cdots=x_{l-s}=0\}} 2^{s+t} d\mu(x) \\ &\le C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \sum_{q=0}^{t} \frac{1}{2^l+2^q} \sum_{s=0}^{q} 2^s + C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \frac{2^t(l-t)}{2^l} \\ &\qquad + C \sum_{l=1}^{\infty} \frac{1}{\alpha(l)} \sum_{t=0}^{l-1} \frac{2^t(l-t)^2}{2^l} \end{aligned}$$

$$\leq C\sum_{l=1}^{\infty}\frac{1}{\alpha(l)}.$$

The inequality $\left|D^{\omega}_{n-2^{|n|}}(\tau_{|n|}(x))\right| \leq n-2^{|n|}$ gives

$$\sum_D \leq \sum_{l=1}^{\infty}\frac{1}{\alpha(l)}\sum_{t=0}^{l-1}\int_{J_t^l}\sup_{\substack{|n|=l\\ q<l}}\sup_{|n-2^{|n|}|=q}\frac{n-2^{|n|}}{n}d\mu(x)$$

$$\leq C\sum_{l=1}^{\infty}\frac{1}{\alpha(l)}\sum_{t=0}^{l-1}2^{-t}\leq C\sum_{l=1}^{\infty}\frac{1}{\alpha(l)}.$$

The lower estimation comes from Lemma 2.1.

This completes the proof of Theorem 2.2. □

Let $\alpha\in\mathbf{R}$, and define the nth (C,α) Fejér kernel $K_n^{\phi,\alpha}$ and the weighted maximal function of the (C,α) Fejér kernels $K_\beta^{\phi,\alpha,*}$ by

$$K_n^{\phi,\alpha}:=\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}D_k^\phi,\quad K_\beta^{\phi,\alpha,*}:=\sup_{n\in\mathbf{N}}\frac{|K_n^{\phi,\alpha}|}{\beta([\log n])},$$

where $\phi=\omega$ or κ and $A_n^\alpha:=\frac{(1+\alpha)\dots(n+\alpha)}{n!}$ for any $n\in\mathbf{N},\alpha\in\mathbf{R}(\alpha\neq-1,-2,\dots)$. It is known that $A_n^\alpha\sim n^\alpha$.

To investigate $K_\beta^{\omega,\alpha,*}$, we have to use the following lemma of Gát and Goginava [5]:

Lemma 2.6 (G. Gát, U. Goginava). *Let $\alpha\in(0,1)$ and $n:=n^{(A)}=n_A2^A+\dots+n_02^0$, then*

$$|K_n^{\omega,\alpha}|\leq\frac{c(\alpha)}{n^\alpha}\sum_{i=0}^{A}\left\{\sum_{p=1}^{i}2^{p(\alpha-1)}\sum_{j=2^{p-1}}^{2^p-1}|K_j^\omega|+2^{i\alpha}|K_{2^i-1}^\omega|+2^{i\alpha}D_{2^i}\right\}.$$

Theorem 2.7. *Let $0<\alpha\leq1$, then there are positive absolute constants c,C (c,C depend only on α) such that*

$$c\sum_{A=0}^{\infty}\frac{1}{\beta(A)}\leq\|K_\beta^{\omega,\alpha,*}\|_1\leq C\sum_{A=0}^{\infty}\frac{1}{\beta(A)}.$$

This means that the behavior of the weighted maximal function of the (C,α) kernels is the same as the behavior of the weighted maximal function of the $(C,1)$ kernels with respect to this issue.

Corollary 2.8. $K_\beta^{\omega,\alpha,*}\in L^1$ *if and only if* $\sum_{A=0}^{\infty}\frac{1}{\beta(A)}<\infty$.

Proof. $\alpha=1$ is given by Lemma 2.1.

Let $|n|=A$. Then by Lemma 2.6 of Gát and Goginava we have

$$\frac{|K_n^{\omega,\alpha}|}{\beta(A)}\leq\frac{C(\alpha)}{2^{A\alpha}\beta(A)}\sum_{i=0}^{A}\left\{\sum_{p=1}^{i}2^{p(\alpha-1)}\sum_{j=2^{p-1}}^{2^p-1}|K_j^\omega|+2^{i\alpha}|K_{2^i-1}^\omega|+2^{i\alpha}D_{2^i}\right\}$$

$$\leq\frac{C(\alpha)}{2^{A\alpha}}\sum_{i=0}^{A}\left\{\sum_{p=1}^{i}2^{p(\alpha-1)}\sum_{j=2^{p-1}}^{2^p-1}\frac{|K_j^\omega|}{\beta(p-1)}+2^{i\alpha}\frac{|K_{2^i-1}^\omega|}{\beta(i-1)}+2^{i\alpha}\frac{D_{2^i}}{\beta(i)}\right\}$$

$$\leq C(\alpha)(K_\beta^{\omega,*}+D_\beta^{\omega,*}).$$

This, Lemma 2.1 and [3] of Gát gives that the upper estimation holds for $K_\beta^{\omega,\alpha,*}$.

To make the lower estimation we need to investigate $K_{2^A}^{\phi,\alpha}$, where $\phi = \omega$ or κ.

On the set $I_A \backslash I_{A+1}$ we have

$$\sum_{j=0}^{2^A} A_{2^A-j}^{\alpha-1} D_j^\phi(x) = \sum_{j=0}^{2^A} A_{2^A-j}^{\alpha-1} j = \sum_{l=0}^{2^A} A_l^{\alpha-1}(2^A - l).$$

Therefore by an Abel transformation and $A_{l+1}^{\alpha-1} = A_l^{\alpha-1}\frac{\alpha+l}{l+1} < A_l^{\alpha-1}$ it follows that

$$\begin{aligned}\sum_{l=0}^{2^A} A_l^{\alpha-1}(2^A - l) &= \sum_{l=0}^{2^A-2}(A_l^{\alpha-1} - A_{l+1}^{\alpha-1})\sum_{j=1}^{l}(2^A - j) + A_{2^A-1}^{\alpha-1}\sum_{l=1}^{2^A-1}(2^A - l) \\ &\geq A_{2^A-1}^{\alpha-1}\sum_{l=1}^{2^A-1}(2^A - l) = A_{2^A-1}^{\alpha-1}\frac{2^A(2^A-1)}{2} > 0\end{aligned}$$

and

$$K_{2^A}^{\phi,\alpha}(x) = \frac{1}{A_{2^A}^\alpha}\sum_{j=0}^{2^A} A_{2^A-j}^{\alpha-1} D_j^\phi(x) \geq \frac{1}{A_{2^A}^\alpha} A_{2^A-1}^{\alpha-1}\frac{2^A(2^A-1)}{2}.$$

Thus,

$$\begin{aligned}\|K_\beta^{\phi,\alpha,*}\|_1 &= \sum_{A=0}^{\infty}\int_{I_A\backslash I_{A+1}} K_\beta^{\phi,\alpha,*}(x)d\mu(x) \\ &\geq \sum_{A=0}^{\infty}\int_{I_A\backslash I_{A+1}} \frac{\left|K_{2^A}^{\phi,\alpha}(x)\right|}{\beta(A)}d\mu(x) \\ &\geq \sum_{A=0}^{\infty}\frac{1}{\beta(A)}\int_{I_A\backslash I_{A+1}} \frac{1}{A_{2^A}^\alpha}A_{2^A-1}^{\alpha-1}\frac{2^A(2^A-1)}{2}d\mu(x) \\ &\geq c\sum_{A=0}^{\infty}\frac{1}{\beta(A)}.\end{aligned}$$

This completes the proof of Theorem 2.7. □

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